

THE MOTIVIC t -STRUCTURE FOR RELATIVE 1-MOTIVES

SIMON PEPIN LEHALLEUR

ABSTRACT. We construct and study a candidate for the standard motivic t -structure on the triangulated category $\mathbf{DA}^1(S, \mathbb{Q})$ of relative cohomological 1-motives with rational coefficients over a noetherian finite dimensional scheme S . This t -structure is defined as a generated t -structure, and we show it is non-degenerate. We relate its heart $\mathbf{MM}^1(S)$ with Deligne 1-motives over S ; in particular, when S is regular, the category of Deligne 1-motives embeds in $\mathbf{MM}^1(S)$ fully faithfully. We also study the inclusion of $\mathbf{DA}^1(S)$ into the larger category $\mathbf{DA}^{\mathrm{coh}}(S)$ of relative cohomological motives on S , and prove that its right adjoint ω^1 , the motivic Picard functor, preserves compact objects.

CONTENTS

Introduction	2
Structure of this paper	4
Acknowledgements	4
Background, conventions and notations	4
Homological algebra in abelian and triangulated categories	4
Schemes and group schemes	5
Triangulated categories of motives	5
1. Triangulated categories of n -motives	6
1.1. Definitions	7
1.2. Permanence properties	8
1.3. Continuity	13
1.4. Over a field	13
1.5. Homological vs cohomological motives	16
1.6. Nearby cycles	16
2. Commutative group schemes and motives	19
2.1. Motives of commutative group schemes	19
2.2. Motives of Deligne 1-motives	25
2.3. Picard complex	26
3. Motivic Picard functor	34
3.1. Definition and elementary properties	34
3.2. The functors ω^n over a perfect field	39
3.3. Computation and finiteness of ω^1	41
4. Motivic t -structures	45
4.1. Generated t -structures	45
4.2. The t -structures over a field	52
4.3. Deligne 1-motives and the heart	54
Appendix A. Deligne 1-motives	60
A.1. Definitions	61
A.2. Continuity and smoothness	62
A.3. Pushforward and Weil restriction	63
Appendix B. Motivic cohomology in degrees $(*, \leq 1)$	65
References	69

INTRODUCTION

We now have at our disposal a mature theory of triangulated categories of motivic sheaves with rational coefficients over general base schemes. Given a noetherian finite-dimensional scheme S , there is a tensor triangulated category $\mathbf{DA}(S)$ which satisfies a number of properties which we state informally.

- There are *realisation functors* from $\mathbf{DA}(S)$ to classical triangulated categories of coefficients: derived categories of abelian sheaves for the classical topology in the Betti setting [Ayo10] and derived categories of ℓ -adic sheaves in the ℓ -adic setting [Ayo14a] [CD15].
- The 2-functor $S \mapsto \mathbf{DA}(S)$ has a rich structure leading to a “formalism of Grothendieck operations” [Ayo07a] [Ayo07b] (including nearby and vanishing cycles) which is compatible via realisation functors to the classical Grothendieck operations for sheaves in the Betti setting and for ℓ -adic sheaves in the ℓ -adic setting.
- Morphisms groups in $\mathbf{DA}(S)$ are related to rational algebraic K -theory for S regular [CD, Corollary 14.2.14] and to Bloch’s higher Chow groups when S is smooth over a field [CD, Example 11.23].
- There is a natural finiteness condition leading to a subcategory $\mathbf{DA}_c(S)$ of “constructible” motivic sheaves, which is stable under the six operations and maps to constructible derived categories via realisations functors [Ayo14a, §8] [CD, §15].
- The category $\mathbf{DA}(S)$ can be constructed in several ways, each of which captures important aspects of the theory: motives without transfers [Ayo14a], Beilinson motives $\mathbf{DM}_B(S)$ [CD], motives with transfers $\mathbf{DM}(S)$ [CD], h -motives $\mathbf{DM}_h(S)$ [CD] [CD15], etc.
- In particular, when S is the spectrum of a perfect field k , the category $\mathbf{DA}(k)$ is equivalent to $\mathbf{DM}(k)$, which gives access to Voevodsky’s cancellation theorem [Voe10] and to the theory of homotopy invariant sheaves with transfers [VSF00] [MVW06a].

In view of those achievements, a major remaining question is the existence of the motivic t -structure on $\mathbf{DA}(S)$, which would provide an abelian category of mixed motivic sheaves realising the conjectures of Beilinson [Jan94]. Here is one possible statement in terms of the ℓ -adic realisation.

Conjecture 0.1. *Let S be a noetherian finite-dimensional scheme and ℓ a prime invertible on S .*

- *The ℓ -adic realisation functor $R_\ell : \mathbf{DA}_c(S) \rightarrow D_c^b(S_{\text{ét}}, \mathbb{Q}_\ell)$ is conservative.*
- *There exists a non-degenerate t -structure $t_{\mathbf{MM}}$ on $\mathbf{DA}_c(S)$ such that if we equip $D_c^b(S_{\text{ét}}, \mathbb{Q}_\ell)$ with its standard t -structure, the functor R_ℓ is t -exact.*

Because we assume conservativity, the t -structure $t_{\mathbf{MM}}$ is uniquely determined by the compatibility with R_ℓ if it exists.

The case where S is the spectrum of a field of characteristic 0 is already extremely interesting; the conjecture in that case implies Grothendieck’s standard conjectures on algebraic cycles [Bei12] and the Bloch-Beilinson-Murre conjectures on the structure of Chow groups of smooth projective varieties [Jan94]. Moreover, a theorem of Bondarko [Bon15, Theorem 3.1.4] shows that for a large class of schemes, if $t_{\mathbf{MM}}(K)$ exists for any residue field K of the scheme S , then the perverse analogue ${}^p t_{\mathbf{MM}}(S)$ of $t_{\mathbf{MM}}(S)$ exists (and one can presumably reconstruct the standard motivic t -structure $t_{\mathbf{MM}}(S)$, at least on compact objects, by a gluing argument).

Since the general conjecture seems inaccessible, one looks for subcategories of $\mathbf{DA}(S)$ where one can hope to construct the restriction of the conjectural t -structure. For $n \in \mathbb{N}$, we introduce the subcategory $\mathbf{DA}_n(S)$ of homological n -motives, i.e., the subcategory generated by homological motives of smooth S -schemes of relative dimension $\leq n$. It seems reasonable to conjecture further that $t_{\mathbf{MM}}$ should restrict to a t -structure $t_{\mathbf{MM},n}$ on $\mathbf{DA}_{n,c}(S)$ (unlike its perverse counterpart ${}^p t_{\mathbf{MM}}$). For $n \geq 2$, we have no idea how to construct $t_{\mathbf{MM},n}$ even when S is a field. Our goal is to provide a reasonable candidate for $t_{\mathbf{MM},1}$.

For a perfect field k , the structure of $\mathbf{DA}_1(k)$ and $t_{\mathbf{MM},1}$ has already been extensively studied. Here is a summary of the main results, transferred from the set-up of \mathbf{DM}^{eff} in the original papers to \mathbf{DA} via the cancellation theorem and the comparison theorem of [CD, Corollary 16.2.22] (for details on these results and the translation, we refer the reader to Sections 3.2 and 4.2).

Theorem 0.2 (Voevodsky, Orgogozo [Org04], Barbieri-Viale-Kahn [BVK], Ayoub-Barbieri-Viale [ABV09], Ayoub [Ayo11]). *Let k be a perfect field and ℓ a prime different from $\text{char}(k)$.*

- (i) *There exists a non-degenerate t structure $t_{\mathbf{MM},1}$ on $\mathbf{DA}_1(k)$ which restricts to $\mathbf{DA}_{1,c}(k)$.*
- (ii) *There is an equivalence of t -categories*

$$\mathbf{DA}_{1,c}(k) \simeq D^b(\mathcal{M}_1(k))$$

where $\mathcal{M}_1(k)$ is the abelian category of Deligne 1-motives with rational coefficients over k [Del74].

- (iii) *The ℓ -adic realisation functor $R_\ell : \mathbf{DA}_{1,c}(k) \rightarrow D(k, \mathbb{Q}_\ell)$ is conservative and t -exact.*
- (iv) *The inclusion of $\mathbf{DA}_1(k)$ into the category $\mathbf{DA}_{\text{hom}}(k)$ of “homological” motives admits a left adjoint, the “motivic Albanese functor” $\mathbf{LAlb} : \mathbf{DA}_{\text{hom}}(k) \rightarrow \mathbf{DA}_1(k)$, which sends compact objects to compact objects, and whose value on the motive of a smooth k -variety X is closely related to its Albanese variety.*

Our work builds on these results and the six operations formalism to produce a similar picture for $\mathbf{DA}_1(S)$.

The most natural approach to a motivic t -structure on $\mathbf{DA}_1(S)$ would proceed by combining the t -structures on $\mathbf{DA}_1(s)$ provided by the previous theorem for all points s of S to a t -structure on $\mathbf{DA}_1(S)$, i.e., proving that the subcategories $\mathbf{DA}_1(S)_{\geq 0} := \{M \in \mathbf{DA}_1(S) \mid \forall s \in S, s^*M \in \mathbf{DA}_1(s)_{\geq 0}\}$ and $\mathbf{DA}_1(S)_{\leq 0} := \{M \in \mathbf{DA}_1(S) \mid \forall s \in S, s^*M \in \mathbf{DA}_1(s)_{\leq 0}\}$ form a t -structure, which would then automatically be compatible with standard t -structures on target categories of realisation functors when they are defined. We do not now how to prove this in this generality, even when restricting to subcategories of compact objects; the gluing arguments of [Bon15, §3.2] are tailored for “perverse” t -structures and cannot be applied directly.

We thus implement an alternative approach, which is inspired by another description [Ayo11, Proposition 3.7] of $t_{\mathbf{MM},1}(k)$ for a perfect field k , as a generated t -structure in the sense of [Ayo07a, Definition 2.1.71]. This leads to a t -structure $t_{\mathbf{MM},1}(S)$ on $\mathbf{DA}_1(S)$ (Definition 4.9). Let us write $\mathbf{MM}_1(S)$ for its heart.

Theorem 0.3 (4.28, 4.22, 4.30, 4.23). *Let S be a noetherian finite-dimensional scheme.*

- (i) *If S is the spectrum of a field k , the t -structure $t_{\mathbf{MM},1}(k)$ coincides with the t -structure of Theorem 0.2.*
- (ii) *The t -structure $t_{\mathbf{MM},1}$ is non-degenerate.*
- (iii) *Write $\mathcal{M}_1(S)$ for the \mathbb{Q} -linear category of Deligne 1-motives over S with rational coefficients. The natural functor $\Sigma^\infty : \mathcal{M}_1(S) \rightarrow \mathbf{DA}_1(S)$ factors through $\mathbf{MM}_1(S)$, and is fully faithful if S is regular.*
- (iv) *Let G be a smooth commutative group scheme with connected fibers. Then the motive $\Sigma^\infty G_{\mathbb{Q}}[-1]$ is in $\mathbf{MM}_1(S)$.*

The result on $\Sigma^\infty G_{\mathbb{Q}}[-1]$ was announced in [AHPL14]; there, this motive appeared as a graded piece in a “Künneth-type” decomposition of the homological motive $M_S(G)$ [AHPL14, Theorem 3.3] which maps to the relative first homology sheaf of G in realisations.

In the relative situation, it is unclear whether the left adjoint \mathbf{LAlb} of the inclusion $\mathbf{DA}_1(S) \rightarrow \mathbf{DA}_{\text{hom}}(S)$ actually exists. We can however define a motivic analogue of the *Picard scheme*. We have a category $\mathbf{DA}^1(S)$ of *cohomological 1-motives* (resp. $\mathbf{DA}^{\text{coh}}(S)$ of cohomological motives) and it turns out that $\mathbf{DA}^1(S) = \mathbf{DA}_1(S)(-1)$ (Proposition 1.27), so that $\mathbf{DA}^1(S)$ also has a motivic t -structure $t_{\mathbf{MM}}^1 = t_{\mathbf{MM},1}(-1)$ which satisfies analogues of the theorems above. The inclusion $\mathbf{DA}^1(S) \rightarrow \mathbf{DA}^{\text{coh}}(S)$ admits a right adjoint $\omega^1 : \mathbf{DA}^{\text{coh}}(S) \rightarrow \mathbf{DA}^1(S)$, as a corollary of Neeman’s version of Brown representability for compactly generated categories; the interesting fact is that, unlike most functors constructed this way, ω^1 satisfies a strong finiteness property.

Theorem 0.4 (3.19). *Let S be a noetherian finite-dimensional excellent scheme satisfying resolution of singularities by alterations. Then ω^1 sends compact objects to compact objects.*

The main step of the proof is to compute ω^1 in a special case, namely $\omega^1(f_*\mathbb{Q}_X)$ with S regular and $f : X \rightarrow S$ smooth projective “Pic-smooth” (Definition 2.24). In this case, Theorem 3.14 shows that

$$\omega^1(f_*\mathbb{Q}_X) \simeq \Sigma^\infty P(X/S)(-1)[-2]$$

where $P(X/S)$ is the *Picard complex* of f , an object closely related to the Picard scheme of X over S .

The results above on $t_{\mathbf{MM},1}$ and ω^1 also have (simpler) counterparts for the category $\mathbf{DA}_0(S)$ of 0-motives which we establish along the way.

The main question which this work leaves open is whether the t -structure $t_{\mathbf{MM},1}$ restricts to compact objects and whether the resulting t -structure on $\mathbf{DA}_{1,c}(S)$ satisfies the analogue of Conjecture 0.1, i.e., whether the ℓ -adic realisation on $\mathbf{DA}_{1,c}(S)$ is then t -exact. This will be established in the case where S is of characteristic 0 in subsequent work.

Structure of this paper. Let S be a finite dimensional noetherian scheme. In Section 1, we introduce the categories $\mathbf{DA}_n(S)$ of homological n -motives (resp. $\mathbf{DA}^n(S)$ of cohomological n -motives) which are full subcategories of $\mathbf{DA}(S)$ generated as triangulated categories with small sums by homological (resp. cohomological) motives of smooth (resp. proper) S -schemes of relative dimension less than n (Definition 1.1). We then study their permanence properties under the six operations (Propositions 1.10 to 1.17) and prove that the homological and cohomological variants are closely related (Proposition 1.27).

In Section 2, we study the motives attached to smooth commutative group schemes over S and prove that they live in $\mathbf{DA}_1(S)$ (Proposition 2.11). We also study motives attached to Deligne 1-motives. Finally, we introduce a motive attached to what we call the Picard complex $P(X/S)$ of a morphism of schemes $f : X \rightarrow S$. It is an object in a derived category of sheaves which packages together information about the relative connected components of f and the Picard scheme of X/S ; under certain hypotheses, this yields a motive in $\mathbf{DA}_{1,c}(S)$ (Corollary 2.33).

In Section 3, we introduce and study the right adjoint $\omega^1 : \mathbf{DA}^{\mathrm{coh}}(S) \rightarrow \mathbf{DA}^1(S)$ to the embedding of cohomological 1-motives into cohomological motives. We first establish a number of relatively formal results involving its commutation properties with the six operations (Proposition 3.3). The main result is then that ω^1 preserves constructibility (Theorem 3.19). This relies on combining techniques from [AZ12] with a computation of $\omega^1(f_*\mathbb{Q}_X)$ in a favorable situation: the precise statement involves the motive of the Picard complex from the previous section.

In Section 4, we finally introduce a candidate for the motivic t -structure on $\mathbf{DA}_1(S)$ and $\mathbf{DA}^1(S)$, using the formalism of generated t -structures. A number of equivalent generating families can be used for this purpose (see Definition 4.3). We prove some basic exactness properties for the six operations. The main result we show is that motives attached to Deligne 1-motives lie in the heart $\mathbf{MM}_1(S)$, and that more precisely the category $\mathcal{M}_1(S)$ embeds fully faithfully into $\mathbf{MM}_1(S)$ for S regular.

Appendix A provide technical results about Deligne 1-motives over a general base. Appendix B gathers some computations of motivic cohomology groups for $\mathbb{Q}(0)$ and $\mathbb{Q}(1)$ which are used at several places in the text.

ACKNOWLEDGEMENTS

This work is based on the main part of my PhD thesis, done under the supervision of Joseph Ayoub at the University of Zürich. I would like to thank him dearly for his constant support, both mathematical and personal.

My thesis was reviewed by Annette Huber and Mikhail Bondarko, and I thank them for their comments. I would also like to thank Giuseppe Ancona, Luca Barbieri-Viale, Javier Fresan, Annette Huber, Shane Kelly, Andrew Kresch, Michel Raynaud and Alberto Vezzani for discussions or e-mail exchanges on the topic of this paper.

BACKGROUND, CONVENTIONS AND NOTATIONS

We collect here several conventions and pieces of notations which will be used throughout this paper.

Homological algebra in abelian and triangulated categories. When discussing complexes in abelian categories and t -structures on triangulated categories, we consistently use homological indexing conventions.

Let $F : \mathcal{T} \rightarrow \mathcal{T}'$ be a triangulated functor between triangulated categories with t -structures. We say that F is t -positive (resp. t -negative) if $F(\mathcal{T}_{\geq 0}) \subset \mathcal{T}'_{\geq 0}$ (resp. $F(\mathcal{T}_{\leq 0}) \subset \mathcal{T}'_{\leq 0}$).

Let \mathcal{T} be a triangulated category, and \mathcal{G} a family of objects of \mathcal{T} . We introduce a number of subcategories of \mathcal{T} generated in various ways by \mathcal{G} . We denote by $\langle \mathcal{G} \rangle$ (resp. $\langle \mathcal{G} \rangle_+$, $\langle \mathcal{G} \rangle_-$) the

smallest triangulated subcategory of \mathcal{T} stable by direct factors (resp. the smallest subcategory stable by extensions, positive shifts and direct factors, the smallest subcategory stable by extensions, negative shifts and direct factors) containing \mathcal{G} . Assume now that \mathcal{T} admits small sums. We denote by $\ll \mathcal{G} \gg$ (resp. $\ll \mathcal{G} \gg_+$, $\ll \mathcal{G} \gg_-$) the smallest triangulated subcategory of \mathcal{T} (resp. the smallest subcategory stable by extensions, small sums and $[+1]$, the smallest subcategory stable by extensions, small sums and $[-1]$) containing \mathcal{G} . Note that $\langle \mathcal{G} \rangle \subset \ll \mathcal{G} \gg$ by [Ayo07a, Lemme 2.1.17].

In the constructions above, we refer informally to \mathcal{G} as the *generating family* to objects of \mathcal{G} as *generators*. In each case, these subcategories can be defined by a (possibly transfinite in the $\ll - \gg$ cases) induction: start with the full subcategory with objects $\mathcal{G}[\mathbb{Z}]$; to pass to a successor ordinal, introduce, depending on the case, cones of all morphisms and direct factors of all objects, just the cones and direct factors, just the cocones and direct factors, the cones, direct factors and small sums, \dots ; finally, to pass to a limit ordinal, take the union over all previous subcategories. These subcategories do not change if one replaces \mathcal{T} by a triangulated subcategory containing \mathcal{G} (and stable under small sums for the $\ll - \gg$ variants), so that we will in general not need to specify the ambient triangulated category, which we omit from the notation.

We adopt the notational convention that functors are derived by default, i.e., we write f_* for Rf_* , f^* for Lf^* , \otimes for \otimes^L , a_{tr} for La_{tr} , etc. In the few cases where we need to refer to the “underived functor”, that is, the underlying Quillen functor at the level of model categories, we underline the notation, i.e., we write \underline{f}_* , \underline{f}^* , $\underline{\otimes}$, $\underline{a}_{\text{tr}}$, etc.

Schemes and group schemes. Unless specified, all schemes are noetherian and finite dimensional, all morphisms of schemes are of finite type, and all smooth morphisms are assumed to be separated. The notation \mathbf{Sm}/S (resp. \mathbf{Sch}/S) denotes the category of all smooth S -schemes (resp. all separated finite type S -schemes), usually considered as a site with the étale topology.

We say that a scheme S admits the resolution of singularities by alterations if for any separated S -scheme X of finite type and any nowhere dense closed subset $Z \subset X$, there is a projective Galois alteration $g : X' \rightarrow X$ with X' regular and such that $g^{-1}(Z)$ is a normal crossing divisor. Recall that by [dJ97, Corollary 5.15], any separated finite type scheme over a noetherian excellent base of dimension ≤ 2 admits the resolution of singularities by alterations.

Let us recall basic terminology and facts about exact sequences of group schemes. Let

$$(C) : 0 \rightarrow G' \xrightarrow{i} G \xrightarrow{p} G'' \rightarrow 0$$

be a sequence of commutative group schemes over a scheme S . We say that (C) is exact if it induces an exact sequence of fppf sheaves on \mathbf{Sch}/S . If (C) is exact, then G' is the scheme-theoretic kernel of p and p is a surjective morphism of schemes. In the other direction, if p is an fppf morphism and G' is its scheme-theoretic kernel, then (C) is exact. Moreover, if the group schemes involved are smooth, then one obtains an equivalent definition (and results) by replacing the fppf topology with the étale topology.

Triangulated categories of motives. We work in most cases in the context of the stable homotopical 2-functor $\mathbf{DA}^{\text{ét}}(-, \mathbb{Q})$ considered in [Ayo14a, §3].

Since we only consider the étale topology and rational coefficients, we simplify the notation and write $\mathbf{DA}(S)$ for $\mathbf{DA}^{\text{ét}}(S, \mathbb{Q})$. The category $\mathbf{DA}(S)$ is equivalent to several other constructions of triangulated categories of motives with rational coefficients, e.g. Beilinson motives [CD]: see [CD, §16] for various comparison theorems.

By [Ayo07a], the functor $\mathbf{DA}(-)$ admits the functoriality of the Grothendieck six operations. In particular, for any quasi-projective morphism $f : S \rightarrow T$, Ayoub constructs adjoint pairs

$$\begin{aligned} f^* : \mathbf{DA}(T) &\rightleftarrows \mathbf{DA}(S) : f_* \\ f_! : \mathbf{DA}(S) &\rightleftarrows \mathbf{DA}(T) : f^! \end{aligned}$$

and when f is smooth

$$f_{\#} : \mathbf{DA}(S) \rightleftarrows \mathbf{DA}(T) : f^*.$$

There is a morphism of functors $f_! \rightarrow f_*$, which is an isomorphism for f projective.

Note that for those operations, as well as for the pullbacks and pushforwards functors on derived categories of sheaves on $\mathbf{Sm}/-$, the notation f^* , f_* , \dots stands for the triangulated or derived functor. When we want to use the underived functor, we underline the functor: \underline{f}^* , \underline{f}_* , \dots .

In the definitions of the Grothendieck operations, one can relax the condition f quasi-projective in the following ways.

- (i) As observed in [Ayo15, Appendice 1.A], one can define f^* and f_* for any morphism f (without any finiteness hypothesis), and prove for instance that proper base change [Ayo14a, Proposition 3.5], the $\mathrm{Ex}_\#^*$ isomorphism [Ayo14a, Proposition 3.6] and “regular base change” [Ayo15, Corollaire 1.A.4] still hold.
- (ii) As observed in [CD, Theorem 2.2.14], one can define the exceptional functors $f_!$ and $f^!$ for any f separated of finite type, and prove that all the properties in [Ayo07a] still hold (with in particular $f_! \simeq f_*$ for any f proper).

We freely use these more general constructions and results.

The six operations for $\mathbf{DA}(-)$ satisfy a large number of properties and compatibilities (see [Ayo14a, Proposition 3.2], [Ayo07a, Scholie 1.4.2]). For results which come up repeatedly in this thesis, we introduce the following terminology. Let

$$\begin{array}{ccc} Z & \xrightarrow{\tilde{g}} & X \\ \tilde{f} \downarrow & & \downarrow f \\ W & \xrightarrow{g} & Y \end{array}$$

be a cartesian square.

- By the $\mathrm{Ex}_\#^*$ isomorphism (resp. the $\mathrm{Ex}_*^!$ isomorphism, the $\mathrm{Ex}_!^*$ isomorphism), we mean the natural isomorphism $\tilde{f}_\# \tilde{g}^* \xrightarrow{\sim} g^* f_\#$ for f smooth (resp. the natural isomorphism $\tilde{f}_* \tilde{g}^! \xrightarrow{\sim} g^! f_*$, the natural isomorphism $g^* f_! \xrightarrow{\sim} \tilde{f}_! \tilde{g}^!$).
- By “smooth base change”, we mean the natural isomorphism $\tilde{f}_* \tilde{g}^* \xrightarrow{\sim} g^* f_*$ for g smooth.
- By “proper base change”, we mean the natural isomorphism $g^* f_* \xrightarrow{\sim} \tilde{f}_* \tilde{g}^*$ for f proper.
- Let $i : Z \rightarrow X$ be a closed immersion and $j : U \rightarrow X$ be the complementary open immersion. When we write “by localisation”, we mean the use of the distinguished triangle of functors

$$j_\# j^* \rightarrow \mathrm{id} \rightarrow i_* i^* \xrightarrow{+}.$$

Dually, when we write “by colocalisation”, we mean the use of the distinguished triangle of functors

$$i_! i^! \rightarrow \mathrm{id} \rightarrow j_* j^* \xrightarrow{+}.$$

- By “relative purity”, we mean the fact that for any smooth morphism $f : S \rightarrow T$ of relative dimension d , there are isomorphism of functors $f_! \simeq f_\#(d)[2d]$ and $f^! \simeq f^*(-d)[-2d]$.
- By “separation” or “the separation property for \mathbf{DA} ”, we mean the fact that for any surjective morphism of finite type (resp. any finite surjective radicial morphism) $f : S \rightarrow T$, the functor $f^* : \mathbf{DA}(T) \rightarrow \mathbf{DA}(S)$ is conservative (resp. an equivalence of categories) [Ayo14a, Theoreme 3.9].
- By “absolute purity”, we mean the fact that for any closed immersion $i : S \rightarrow T$ of codimension d with S, T regular schemes, we have $i^! \mathbb{Q}_T \simeq \mathbb{Q}_S(-d)[-2d]$ ([Ayo14a, Corollaire 7.5] and [Ayo14a, Remarque 11.2]).
- By “cohomological h -descent”, we mean the fact that for any finite type morphism $f : S \rightarrow T$ of schemes and any hypercovering $\pi : S_\bullet \rightarrow S$ in Voevodsky’s h -topology, [CD, Theorem 14.3.4], the natural morphism of functors

$$f_* f^*(-) \rightarrow f_* \pi_* \pi^* f^*(-)$$

(which is part of the algebraic derivator structure for $\mathbf{DA}(-)$) is an isomorphism. More precisely, we always use this through the induced descent spectral sequence for morphisms groups in $\mathbf{DA}(-)$.

1. TRIANGULATED CATEGORIES OF N-MOTIVES

Categories of motives are naturally filtered by the dimension of “geometric generators”, and such filtrations have been studied in various motivic contexts [Bei02] [Ayo11] [ABV09]. We give definitions in the context of $\mathbf{DA}(-)$ and prove a number of basic results. Since such a treatment does not appear in the litterature, we provide more than is strictly necessary for the rest of the

paper; outside of this section, we are concerned with the special case of (co)homological 0- and 1-motives.

1.1. Definitions. We fix a base scheme S and an integer $n \geq 0$ for the remainder of this section.

Definition 1.1. The category $\mathbf{DA}^{\text{coh}}(S)$ (resp. $\mathbf{DA}_{\text{hom}}(S)$) of *cohomological motives* (resp. *homological motives*) is the full subcategory of $\mathbf{DA}(S)$ defined as

$$\mathbf{DA}^{\text{coh}}(S) = \ll f_* \mathbb{Q}_X \mid f : X \rightarrow S \text{ proper morphism} \gg$$

(resp.

$$\mathbf{DA}_{\text{hom}}(S) = \ll f_{\sharp} \mathbb{Q}_X \mid f : X \rightarrow S \text{ smooth morphism} \gg).$$

The category $\mathbf{DA}^n(S)$ (resp. $\mathbf{DA}_n(S)$) of *cohomological n -motives* (resp. *homological n -motives*) is the full subcategory of $\mathbf{DA}(S)$ defined as

$$\mathbf{DA}^n(S) = \ll f_* \mathbb{Q}_X \mid f : X \rightarrow S \text{ proper morphism of relative dimension} \leq n \gg$$

(resp.

$$\mathbf{DA}_n(S) = \ll f_{\sharp} \mathbb{Q}_X \mid f : X \rightarrow S \text{ smooth morphism of relative dimension} \leq n \gg).$$

Remark 1.2. As we will see in Proposition 1.27, the categories $\mathbf{DA}_n(S)$ and $\mathbf{DA}^n(S)$ are in fact equivalent as triangulated categories, so that many questions about $\mathbf{DA}^n(S)$ can be reduced to $\mathbf{DA}_n(S)$. In the special cases $n = 0, 1$, this is a crucial ingredient for several results in this paper. However to establish Proposition 1.27 we need some preliminary results which we obtain by studying \mathbf{DA}_n and \mathbf{DA}^n in parallel.

We have subcategories of smooth and geometrically smooth objects. Recall that an object X in a symmetric monoidal category is said to be strongly dualizable if there exists an object X^* together with morphisms $\epsilon : \mathbb{1} \rightarrow X \otimes X^*$ and $\eta : X \otimes X^* \rightarrow \mathbb{1}$ satisfying the classical adjunction triangle laws.

Definition 1.3. The category $\mathbf{DA}^{\text{gsm}}(S)$ (resp. $\mathbf{DA}_{\text{gsm}}^{\text{coh}}(S)$, $\mathbf{DA}_{\text{hom}}^{\text{gsm}}(S)$) of *geometrically smooth motives* (resp. *geometrically smooth cohomological motives* resp. *geometrically smooth homological motives*) is the full subcategory of $\mathbf{DA}(S)$ defined as

$$\mathbf{DA}^{\text{gsm}}(S) = \ll f_{\sharp} \mathbb{Q}_X(-n) \mid f : X \rightarrow S \text{ proper smooth morphism, } n \in \mathbb{Z} \gg$$

(resp.

$$\mathbf{DA}_{\text{gsm}}^{\text{coh}}(S) = \ll f_* \mathbb{Q}_X \mid f : X \rightarrow S \text{ proper smooth morphism} \gg,$$

$$\mathbf{DA}_{\text{hom}}^{\text{gsm}}(S) = \ll f_{\sharp} \mathbb{Q}_X \mid f : X \rightarrow S \text{ proper smooth morphism} \gg).$$

The category $\mathbf{DA}_{\text{sm}}(S)$ (resp. $\mathbf{DA}_{\text{sm}}^{\text{coh}}(S)$, $\mathbf{DA}_{\text{hom}}^{\text{sm}}(S)$) of *smooth motives* (resp. *smooth cohomological motives*, *smooth homological motives*) is defined as

$$\mathbf{DA}_{\text{sm}}(S) = \ll M \in \mathbf{DA}(S) \mid M \text{ strongly dualizable} \gg$$

(resp.

$$\mathbf{DA}_{\text{sm}}^{\text{coh}}(S) = \ll M \in \mathbf{DA}^{\text{coh}}(S) \mid M \text{ strongly dualizable in } \mathbf{DA}(S) \gg,$$

$$\mathbf{DA}_{\text{hom}}^{\text{sm}}(S) = \ll M \in \mathbf{DA}_{\text{hom}}(S) \mid M \text{ strongly dualizable in } \mathbf{DA}(S) \gg).$$

We then define

$$\mathbf{DA}_{\text{gsm}}^n(S) = \mathbf{DA}^n(S) \cap \mathbf{DA}_{\text{gsm}}^{\text{coh}}(S)$$

$$\mathbf{DA}_n^{\text{gsm}}(S) = \mathbf{DA}_n(S) \cap \mathbf{DA}_{\text{hom}}^{\text{gsm}}(S)$$

$$\mathbf{DA}_{\text{sm}}^n(S) = \mathbf{DA}_n(S) \cap \mathbf{DA}_{\text{sm}}^{\text{coh}}(S)$$

$$\mathbf{DA}_n^{\text{sm}}(S) = \mathbf{DA}_n(S) \cap \mathbf{DA}_{\text{hom}}^{\text{sm}}(S)$$

Remark 1.4. In the definition of $\mathbf{DA}_{\text{gsm}}^n(S)$ and $\mathbf{DA}_n^{\text{gsm}}(S)$, we make the choice not to impose the geometric smoothness to “come from” generators of relative dimension $\leq n$. This more restrictive definition seems too strong for the study of geometrically smooth 1-motives, as the proof of Corollary 2.13 below shows.

Lemma 1.5. *Geometrically smooth objects are smooth: $\mathbf{DA}^{\text{gsm}}(S) \subset \mathbf{DA}^{\text{sm}}(S)$, $\mathbf{DA}_{\text{hom}}^{\text{gsm}}(S) \subset \mathbf{DA}_{\text{hom}}^{\text{sm}}(S)$, etc.*

Proof. This follows from relative purity and the projection formula, see e.g. [CD15, Lemma 4.2.8]. \square

Remark 1.6. The converse of the above lemma is not known and it is not clear if one should expect it. Informally, when S is a discrete valuation ring, it would mean that a “motive with good reduction” is realisable in the cohomology of a variety with good reduction.

There is a further reasonable definition of a smooth object in $\mathbf{DA}_c(S)$, namely a motive whose realisations have cohomology sheaves which are local systems (in the appropriate sense, e.g. lisse ℓ -adic sheaves). This is conjecturally equivalent to being strongly dualizable.

Proposition 1.25 below shows that when S is the spectrum of a field, any motive is geometrically smooth.

An important property of smooth compact objects is that they satisfy a form of absolute purity.

Proposition 1.7. *Let $i : Z \rightarrow S$ be a regular immersion of codimension c . For $M \in \mathbf{DA}_c^{\text{sm}}(S)$ (i.e., M strongly dualisable), there is a purity isomorphism*

$$i^* M \simeq i^! M(c)[2c]$$

which is functorial in M , in the sense that for any $f : M \rightarrow N \in \mathbf{DA}_c^{\text{sm}}(S)$ the diagram

$$\begin{array}{ccc} i^* M & \xrightarrow{i^*(f)} & i^* N \\ \downarrow & & \downarrow \\ i^! M(c)[2c] & \xrightarrow{i^!(f)(c)[2c]} & i^! N(c)[2c] \end{array}$$

commutes.

Proof. The idea is to use dualisability to reduce to the usual absolute purity property for the unit object. The functor i^* is monoidal, hence preserves strongly dualisable objects and sends strong duals to strong duals. By biduality, this provides a natural isomorphism

$$i^* M \xrightarrow{\sim} \underline{\text{Hom}}(i^* M^\vee, \mathbb{Q}_Z)$$

By absolute purity, this last group is isomorphic to $\underline{\text{Hom}}(i^* M^\vee, i^! \mathbb{Q}_Z(c)[2c])$, which is itself naturally isomorphic to $i^! \underline{\text{Hom}}(M^\vee, \mathbb{Q}(c)[2c])$ by [Ayo07a, Proposition 2.3.55]. Since $\mathbb{Q}(c)[2c]$ is invertible and M is strongly dualisable, $i^! \underline{\text{Hom}}(M^\vee, \mathbb{Q}(c)[2c]) \simeq i^! \underline{\text{Hom}}(M^\vee, \mathbb{Q})(c)[2c] \simeq i^! M(c)[2c]$. The composition gives the required isomorphism. Each step of the construction is functorial in M . \square

Lemma 1.8. *Let \mathcal{T} be one of $\mathbf{DA}_{\text{hom}}(S)$, $\mathbf{DA}^{\text{coh}}(S)$, $\mathbf{DA}_n(S)$, $\mathbf{DA}^n(S)$ or their subcategories of smooth or geometrically smooth objects. Then the triangulated category \mathcal{T} is compactly generated by its generating family, and an object of \mathcal{T} is compact if and only if it is compact in $\mathbf{DA}(S)$.*

Proof. Write \mathcal{G} for the generating family of \mathcal{T} . By [Ayo14a, Proposition 3.20, Proposition 8.5] and by the fact that strongly dualizable objects in a symmetric monoidal triangulated category are automatically compact, we see that all objects of \mathcal{G} are compact. This means that \mathcal{T} is compactly generated by \mathcal{G} . Write \mathcal{T}_c for the full subcategory of objects of \mathcal{T} which are compact in \mathcal{T} . By [Nee01, Lemma 4.4.5], $\mathcal{T}_c = \langle \mathcal{G} \rangle$. In particular any object of \mathcal{T}_c is compact in $\mathbf{DA}(S)$. The converse implication is obvious. \square

Definition 1.9. We write $\mathbf{DA}_c^{\text{coh}}(S)$, $\mathbf{DA}_{\text{hom},c}(S)$, etc. for the full subcategories of compact objects of $\mathbf{DA}^{\text{coh}}(S)$, $\mathbf{DA}_{\text{hom}}(S)$, etc.

1.2. Permanence properties. The subcategories we have introduced are each stable under certain Grothendieck operations. We start with the compatibilities with the monoidal structure.

Proposition 1.10. *Let S be a base scheme.*

- (i) $\mathbf{DA}^{\text{coh}}(S)$ is stable by tensor products and negative Tate twists.
- (ii) For all $m, n \geq 0$, we have $\mathbf{DA}^m(S) \otimes \mathbf{DA}^n(S) \subset \mathbf{DA}^{m+n}(S)$.
- (iii) For all $m, n \geq 0$, we have $\mathbf{DA}^m(S)(-n) \subset \mathbf{DA}^{m+n}(S)$.
- (iv) $\mathbf{DA}_{\text{hom}}(S)$ is stable by tensor products and positive Tate twists.
- (v) For all $m, n \geq 0$, we have $\mathbf{DA}_m(S) \otimes \mathbf{DA}_n(S) \subset \mathbf{DA}_{m+n}(S)$.
- (vi) For all $m, n \geq 0$, we have $\mathbf{DA}_m(S)(n) \subset \mathbf{DA}_{m+n}(S)$.

The same properties hold for the smooth and geometrically smooth versions of those subcategories.

Proof. First, note that \otimes commutes with small sums in both variables, being a left adjoint. This reduces the proof to checking the result for generators.

Let us prove point (i). Recall that we have a projection formula for $f_!$ and f^* from [Ayo07a, Theoreme 2.3.40], i.e., for any finite type separated morphism $f : S \rightarrow T$ and any $M \in \mathbf{DA}(S), N \in \mathbf{DA}(T)$, we have a natural isomorphism

$$f_!(M \otimes f^*N) \simeq f_!M \otimes N.$$

Let $g : X \rightarrow S$ and $h : Y \rightarrow S$ be proper morphisms. Let $Z = X \times_S Y$ and let $g' : Z \rightarrow Y$ and $h' : Z \rightarrow X$ be the two projections. We have a sequence of isomorphisms

$$\begin{aligned} g_*\mathbb{Q}_X \otimes h_*\mathbb{Q}_Y &\simeq g_!\mathbb{Q}_X \otimes h_!\mathbb{Q}_Y \\ &\simeq g_!(\mathbb{Q}_X \otimes g^*h_!\mathbb{Q}_Y) \\ &\simeq g_!h'_!(g')^*\mathbb{Q}_Y \\ &\simeq g_*h'_*\mathbb{Q}_Z \end{aligned}$$

where the first and fourth isomorphisms follows from properness, the second is the projection formula and the third is the $\mathrm{Ex}_!^*$ isomorphism. This shows that $g_*\mathbb{Q}_X \otimes h_*\mathbb{Q}_Y$ is cohomological. The negative Tate twist $\mathbb{Q}_S(-n)$ is cohomological, as it is a direct factor of $(\mathbb{P}_S^n \rightarrow S)_*\mathbb{Q}$. This finishes the proof of (i). The same proof, combined with the fact that relative dimension is stable by base change and adds up in compositions, gives (ii) and (iii).

For the proof of point (iv), we use a parallel argument; we combine the projection formula for f_\sharp and f^* of [Ayo07b, Proposition 4.5.17] with the Ex_\sharp^* isomorphism and the fact that $\mathbb{Q}_S(n)$ is a direct factor of $(\mathbb{P}_S^n \rightarrow S)_\sharp\mathbb{Q}$ by the projective bundle formula. The same proof, combined with the fact that relative dimension is stable by base change and adds up in compositions, gives (v) and (vi).

Finally, the analogous statement for smooth and geometrically smooth versions follow from similar arguments together with the fact that a tensor product of strongly dualizable objects is strongly dualizable. \square

Proposition 1.11. *Let $f : S \rightarrow T$ be a morphism of schemes. The following operations preserve the subcategories $\mathbf{DA}^{\mathrm{coh}}(-)$.*

- (i) f^* for any f .
- (ii) f_* when f is separated of finite type and S admits the resolution of singularities by alterations.
- (iii) $f_!$ when f is separated of finite type.
- (iv) $f^!$ when f is quasi-finite separated and S admits the resolution of singularities by alterations.

Moreover, they also preserve $\mathbf{DA}_c^{\mathrm{coh}}(-)$ (with the assumption that the schemes involved are excellent for points (ii)-(iv)).

Proof. The results for $\mathbf{DA}_c^{\mathrm{coh}}(-)$ follow from the ones for $\mathbf{DA}^{\mathrm{coh}}(-)$ together with the constructibility theorem for Grothendieck operations [Ayo14a, Theoreme 8.10] and Lemma 1.8. We thus focus on $\mathbf{DA}^{\mathrm{coh}}(-)$. We prove the results in a slightly different order than in the statement: we first establish (i), (iii) (which contains the special case of (ii) for proper morphisms), (iv) for closed immersions, (ii) and finally (iv) in all generality. In each case, we first check that the functor commutes with small sums, and then compute its action on generators of $\mathbf{DA}^{\mathrm{coh}}(-)$.

Proof of (i): the functor f^* is a left adjoint hence commutes with small sums. Moreover proper base change implies that f^* sends generators of $\mathbf{DA}^{\mathrm{coh}}(T)$ to generators of $\mathbf{DA}^{\mathrm{coh}}(S)$.

Proof of (iii): the functor $f_!$ is a left adjoint hence commutes with small sums. Let $g : X \rightarrow S$ be a proper morphism. We need to show that $f_!g_*\mathbb{Q}_X \simeq (f \circ g)_!\mathbb{Q}_X$ is in $\mathbf{DA}^{\mathrm{coh}}(T)$. Because f is assumed to be separated of finite type, the same holds for $f \circ g$. Nagata's theorem [Nag63] [Con07] implies that $f \circ g$ admits a compactification, i.e., that there exists a factorisation $f \circ g = \bar{f} \circ j$ with $j : X \rightarrow \bar{X}$ an open immersion and $\bar{f} : \bar{X} \rightarrow T$ a proper morphism. Let $i : Z \rightarrow \bar{X}$ be a

complementary closed immersion to j . By localisation, we have a distinguished triangle

$$j_! \mathbb{Q}_X \rightarrow \mathbb{Q}_{\bar{X}} \rightarrow i_! \mathbb{Q}_Z \xrightarrow{+}$$

which after applying $\bar{f}_* \simeq \bar{f}_!$ yields

$$\bar{f}_* j_! \mathbb{Q}_X \simeq f_! g_* \mathbb{Q}_X \rightarrow \bar{f}_! \mathbb{Q}_{\bar{X}} \rightarrow (\bar{f}i)_! \mathbb{Q}_Z \xrightarrow{+}.$$

By definition, the second and third terms in this triangle are in $\mathbf{DA}^{\text{coh}}(T)$. This implies that the first is as well.

Proof of (iv) for $f = i$ closed immersion:

The functor $i^!$ has a left adjoint $i_!$ which sends compact objects to compact objects by [Ayo14a, Proposition 8.5]. By [Ayo07a, Lemme 2.1.28] this implies that $i^!$ commutes with small sums.

The blueprint for this proof is taken from Section 2.2.2 of [Ayo07a].

Lemma 1.12, applied to $i : S \rightarrow T$, shows that it is enough to show that, for any $g : X \rightarrow T$ with X connected regular and $g^{-1}(S)$ equal to either X or a normal crossing divisor, the motive $i^! g_* \mathbb{Q}_X$ is compact. Form the cartesian square

$$\begin{array}{ccc} Y & \xrightarrow{i'} & X \\ g' \downarrow & & \downarrow g \\ S & \xrightarrow{i} & T. \end{array}$$

We have an $\text{Ex}_*^!$ isomorphism $i^! g_* \mathbb{Q}_X \simeq g'_* i'^! \mathbb{Q}_X$. By point (iii), it is enough to show that $i'^! \mathbb{Q}_X$ is in $\mathbf{DA}^{\text{coh}}(X)$. By assumption, Y is either equal to X or is a normal crossing divisor; only the second case requires a proof. By [Ayo07a, Lemme 2.2.31] applied to the branches and point (iii) for closed immersions, we reduce to the case of a regular immersion, which then follows from absolute purity and Proposition 1.10 (i).

Proof of (ii):

Using Nagata's theorem and the proper case of point (iii), we reduce to show that $j_* \mathbb{Q}_S$ is in $\mathbf{DA}^{\text{coh}}(T)$ for $j : S \rightarrow T$ open immersion. This now follows from colocalisation and point (iv) for the complementary closed immersion.

Proof of (iv) for f quasi-finite general:

By the same argument as above, using the $\text{Ex}_*^!$ isomorphism, it is enough to show that $f^! \mathbb{Q}_T$ is in $\mathbf{DA}^{\text{coh}}(S)$. Using Zariski's main theorem [Gro66, Théorème 8.12.6], the fact that $j^! \simeq j^*$ for j open immersion, and by point (i) we are reduced to the case of finite morphisms.

If f is finite étale, then $f^! \simeq f^*$ again and we are done by point (i). If f is finite and purely inseparable, then a corollary of the separation property of \mathbf{DA} is that $f^! \simeq f^*$ is an equivalence of categories [Ayo07a, Corollaire 2.1.164]. In general, we proceed by induction on the dimension of T . The proof for the 0-dimensional case follows the same pattern as the inductive step, so we treat both in parallel. If T is 0-dimensional, or generically on T , say above an everywhere dense open set $j : U \rightarrow T$, f is the composite of a finite étale morphism (possibly empty) followed by a finite purely inseparable morphism. Let $l : V \rightarrow S$ be $j \times_T S$ and $k : W \rightarrow S$ be a complementary closed immersion (take W empty in the 0-dimensional case). Then $l^! f^! \mathbb{Q}_T \simeq f_U^! \mathbb{Q}_U$ is in $\mathbf{DA}^{\text{coh}}(V)$ by combining the arguments for finite étale and finite purely inseparable morphisms above. By point (ii), we get that $l_* l^* f^! \mathbb{Q}_T$ is in $\mathbf{DA}^{\text{coh}}(S)$. This concludes the proof for $\dim(T) = 0$. In general, by induction on the dimension and point (iii), we get that $k_! k^! f^! \mathbb{Q}_T$ lies in $\mathbf{DA}^{\text{coh}}(S)$. The colocalisation triangle then shows that $f^! \mathbb{Q}_T$ lies in $\mathbf{DA}^{\text{coh}}(S)$. This completes the proof. \square

Lemma 1.12. *Let S be a scheme admitting resolution of singularities by alterations, $f : X \rightarrow S$ a finite type morphism and $T \subset X$ closed. Then $\mathbf{DA}^{\text{coh}}(X)$ is compactly generated by motives of the form $g_* \mathbb{Q}_{X'}$ with $g : X' \rightarrow X$ a projective morphism and X' connected regular and $g^{-1}(T)$ equal either to X' or to a normal crossing divisor.*

Proof. The reference [Ayo07a, Proposition 2.2.27], specialized to the \mathbb{Q} -linear, separated, homotopical 2-functor $\mathbf{DA}(-)$ proves a similar statement for the category of constructible objects $\mathbf{DA}_c(S)$ (with added positive Tate twists of the generators, and restriction to quasi-projective morphisms). Once one removes the Tate twists, the restriction to quasi-projective morphisms, and remarks

that Statement Proposition (iii) provides the analogue of Corollaire 2.2.21 from loc.cit, the same argument applies verbatim. \square

Proposition 1.13. *Let $f : S \rightarrow T$ be a morphism of schemes. The following operations preserve the subcategories $\mathbf{DA}_{\text{hom}}(-)$ and $\mathbf{DA}_{\text{hom},c}(-)$.*

- (i) f^* for any f .
- (ii) $f_{\#}$ when f is smooth.
- (iii) $f_!$ for any quasi-finite separated morphism f .

Remark 1.14. In the proof of point (iii), we use results from Sections 1.3 and 1.4. The careful reader can check that we do not use the reference 1.13 (iii) in between. We feel this break from logical order is justified by the commodity of having a clean statement.

Proof. The results about $\mathbf{DA}_{\text{hom},c}(-)$ follow from the ones for $\mathbf{DA}_{\text{hom}}(-)$ together with the constructibility result [Ayo14a, Proposition 8.5] and Lemma 1.8. We thus focus on $\mathbf{DA}_{\text{hom}}(-)$.

Proof of (i): The functor f^* is a left adjoint so commutes with small sums. Moreover the $\text{Ex}_{\#}^*$ isomorphism implies that f^* sends generators of $\mathbf{DA}_{\text{hom}}(T)$ to generators of $\mathbf{DA}_{\text{hom}}(S)$.

Proof of (ii): The functor $f_{\#}$ is a left adjoint so commutes with small sums. The fact that generators are sent to homological motives clearly follows from the definition.

Proof of (iii): The functor $f_!$ is a left adjoint so preserves small sums. Using Zariski's Main theorem [Gro66, Théorème 8.12.6] and (ii), we see that it is enough to treat the case of f finite.

We first do the case of closed immersions. The next lemma is proved using Mayer-Vietoris distinguished triangles.

Lemma 1.15. *Let T be a scheme and $\mathcal{U} = \{j_k : U_k \hookrightarrow T\}_{k=1}^n$ be a finite Zariski open covering of T . Let $M \in \mathbf{DA}(S)$. Then*

$$M \in \mathbf{DA}_{\text{hom}}(S) \iff \text{for all } 1 \leq k \leq n, \text{ we have } j_k^* M \in \mathbf{DA}_{\text{hom}}(S).$$

\square

Let $i : Z \rightarrow X$ be a closed immersion and $g : U \rightarrow Z$ be a smooth morphism. We need to show that $i_* g_{\#} \mathbb{Q}_U \in \mathbf{DA}_{\text{hom}}(X)$. There exists a finite open affine cover $\{U_k = \mathbf{Spec}(A_k)\}_{1 \leq k \leq n}$ of U and a finite open affine cover $\{Z_k = \mathbf{Spec}(R_k)\}_{1 \leq k \leq n}$ of Z with $g(U_k) \subset Z_k$ and such that via $g_k := g|_{U_k}^{Z_k}$, the ring A_k takes the form:

$$A_k = R_k[x_1, \dots, x_{n_k}] / (f_1^k, \dots, f_{c_k}^k)$$

with $\left(\det\left(\frac{\partial f_j^k}{\partial x_k}\right)\right)$ invertible in A_k (i.e. g_k is a standard smooth map). We can choose an open affine cover $\{W_k\}$ of X such that $W_k \cap Z = Z_k$. Applying Lemma 1.15 to the open cover W_k and using base change for closed immersions and smooth base change, we can suppose that g itself is a standard smooth map and that $X = \mathbf{Spec}(R)$ is affine.

In this situation, we can lift the equations f_j to $\tilde{f}_j \in R[x_1, \dots, x_n]$. The open set W of X over which the resulting map $\tilde{g} : \mathbf{Spec}(R[x_1, \dots, x_n]/(\tilde{f}_1, \dots, \tilde{f}_n)) \rightarrow X$ is standard smooth contains Z , and \tilde{g} extends g . We have a localisation triangle

$$(W \setminus Z \rightarrow W)_{\#} \tilde{g}_{\#} \mathbb{Q} \rightarrow \tilde{g}_{\#} \mathbb{Q} \rightarrow i_* g_{\#} \mathbb{Q}_U \xrightarrow{+}$$

where the first two terms are in $\mathbf{DA}_{\text{hom}}(X)$. We deduce that $i_* g_{\#} \mathbb{Q}_U \in \mathbf{DA}_{\text{hom}}(X)$ as wanted.

For a general quasi-finite $f : T \rightarrow S$, using localisation, the case of closed immersions and an induction on the dimension of S , we see that we can replace S by any everywhere dense open subset. The case of closed immersion also ensures we can assume S is reduced. By continuity for $\mathbf{DA}_{\text{hom}}(-)$ (proven in Proposition 1.21 below; the proof does not use permanence properties of $\mathbf{DA}_{\text{hom}}(-)$ besides (i)), we see that we can even replace S by any of its generic points. We are thus reduced to the case of a finite field extension, which follows from the following more precise Lemma 1.26 below. \square

Proposition 1.16.

- (i) *Let f be any morphism of schemes. Then f^* preserves the subcategories $\mathbf{DA}^n(-)$ and $\mathbf{DA}_c^n(-)$.*

- (ii) Let $f : S \rightarrow T$ be separated of finite type and of relative dimension m . Then $f_!$ sends $\mathbf{DA}^n(S)$ (resp. $\mathbf{DA}_c^n(S)$) to $\mathbf{DA}^{n+m}(T)$ (resp. $\mathbf{DA}_c^{n+m}(T)$). In particular, if f is quasi-finite, then $f_!$ preserves the subcategories $\mathbf{DA}^n(-)$ and $\mathbf{DA}_c^n(-)$.

Proof. To treat the case of subcategories of compact objects, we combine the following arguments with Lemma 1.8 and the “easy” constructibility result of [Ayo14a, Proposition 8.5]. Consequently, we only treat the case of $\mathbf{DA}^n(-)$.

Statement (i) follows from the fact that f^* , being a left adjoint, commutes with small sums, proper base change and the fact that being of relative dimension $\leq n$ is stable by base change.

The proof of (ii) is the same as that of Proposition 1.11 (iii), keeping track of the relative dimensions involved. \square

Proposition 1.17.

- (i) Let f be any morphism of schemes. Then f^* preserves the subcategories $\mathbf{DA}_n(-)$ and $\mathbf{DA}_{n,c}(-)$.
- (ii) Let $f : S \rightarrow T$ be separated of finite type and of relative dimension m . Then $f_!$ sends $\mathbf{DA}_n(S)$ (resp. $\mathbf{DA}_{n,c}(S)$) to $\mathbf{DA}_{n+m}(T)$ (resp. $\mathbf{DA}_{n+m,c}(T)$). In particular, if f is quasi-finite, then $f_!$ preserves the subcategories $\mathbf{DA}_n(-)$ and $\mathbf{DA}_{n,c}(-)$.

Proof. To treat the case of subcategories of compact objects, we combine the following arguments with the Lemma 1.8 and the “easy” constructibility results of [Ayo14a, Proposition 8.5]. Consequently, we only treat the case of $\mathbf{DA}_n(-)$.

Statement (i) follows from the fact that f^* , being a left adjoint, commutes with small sums, from the $\mathrm{Ex}_\#^*$ isomorphism and the fact that being of relative dimension $\leq n$ is stable by base change.

The proof of (ii) is the same as that of Proposition 1.13 (iii), keeping track of the relative dimensions involved. \square

We list some useful corollaries of the results above.

Corollary 1.18. Let $\mathcal{T}(-)$ be one of $\mathbf{DA}^{\mathrm{coh}}(-)$, $\mathbf{DA}_{\mathrm{hom}}(-)$, $\mathbf{DA}^n(-)$, $\mathbf{DA}_n(-)$ or one of their subcategories of compact objects.

- (i) The system $\mathcal{T}(-)$ localises in the following sense: for $M \in \mathbf{DA}(S)$, $i : Z \rightarrow S$ and $j : U \rightarrow S$ are complementary closed and open immersions, $M \in \mathcal{T}(S)$ if and only if $i^*M \in \mathcal{T}(Z)$ and $j^*M \in \mathcal{T}(U)$.
- (ii) Let $f : T \rightarrow S$ be a finite radicial surjective morphism (e.g. a nil-immersion), $M \in \mathbf{DA}(S)$, $N \in \mathbf{DA}(T)$. Then $M \in \mathcal{T}(S)$ if and only if $f^*M \in \mathcal{T}(T)$, and $N \in \mathcal{T}(T)$ if and only if $f_*N \in \mathcal{T}(S)$.

Proof. Statement (i) follows directly from localisation and the permanence properties above. Similarly, statement (ii) follows directly [Ayo07a, Proposition 2.1.163] (which applies because $\mathbf{DA}(-)$ is separated) and the permanence properties. \square

Finally, let us discuss what happens with internal Homs and duality.

Corollary 1.19. We have $\underline{\mathrm{Hom}}(\mathbf{DA}_{\mathrm{hom},c}(S), \mathbf{DA}_c^{\mathrm{coh}}(S)) \subset \mathbf{DA}_c^{\mathrm{coh}}(S)$. In particular, if S is regular and we take \mathbb{Q}_S as dualizing object, then Verdier duality $\mathbb{D}_S := \underline{\mathrm{Hom}}(-, \mathbb{Q}_S)$ sends compact homological motives to compact cohomological motives.

Proof. If $M \in \mathbf{DA}(S)$ is compact, then $\underline{\mathrm{Hom}}(M, -)$ commutes with small sums. This shows that we can restrict to generators of $\mathbf{DA}^{\mathrm{coh}}(S)$ in the second variable. Using [Nee01, Lemma 4.4.5], we see that we can restrict to generators of $\mathbf{DA}_{\mathrm{hom},c}(S)$ in the first variable. The result then follows from [Ayo07a, Proposition 2.3.51-52], the $\mathrm{Ex}_\#^*$ isomorphism and Proposition 1.11 (ii). \square

Remark 1.20. Even on a regular scheme, the categories of constructible homological and cohomological motives are not anti-equivalent through Verdier duality with dualizing object \mathbb{Q} (see, however, Proposition 1.25 below). Indeed, assume S regular of dimension $d > 0$, let $i : x \rightarrow S$ be the inclusion of a closed point x and $j : U \rightarrow S$ be the complementary open immersion. Then by colocalisation and absolute purity, $j_*\mathbb{Q}_U \in \mathbf{DA}^{\mathrm{coh}}(S)$ sits in a triangle

$$i_*\mathbb{Q}(-d)[-2d] \rightarrow \mathbb{Q}_S \rightarrow j_*\mathbb{Q}_U \xrightarrow{+}.$$

In particular, it is cohomological. On the other hand, we have $\mathbb{D}_S(\mathbb{Q}_S) \simeq \mathbb{Q}_S \in \mathbf{DA}^{\text{coh}}(S)$ and $\mathbb{D}_S(i_! i^! \mathbb{Q}_S) \simeq i_* \mathbb{Q}_S \in \mathbf{DA}^{\text{coh}}(S)$, so that by taking the Verdier dual of the triangle above, we have $\mathbb{D}_S(j_* \mathbb{Q}_U) \in \mathbf{DA}^{\text{coh}}(S)$.

If Verdier duality did exchange homological and cohomological motives, we would have $j_* \mathbb{Q}_U \in \mathbf{DA}_{\text{hom}}(S) \cap \mathbf{DA}^{\text{coh}}(S)$ which is equal to $\mathbf{DA}_0(S)$ by Corollary 3.8 (ii) below. We would then also have $i_* \mathbb{Q}(-d)[-2d] \in \mathbf{DA}_0(S)$; hence, $i^* i_* \mathbb{Q}(-d) \simeq \mathbb{Q}(-d) \in \mathbf{DA}_0(x)$. This is not the case, as can be seen in a number of ways; for instance, in the proof of Corollary 3.8 (iv) we will show that for all $M \in \mathbf{DA}_0(x)$, we have $\text{Hom}(M, \mathbb{Q}(-d)) = 0$.

1.3. Continuity. We have a continuity result for subcategories of compact objects.

Proposition 1.21. *Let I be a cofiltering small category and $(X_i)_{i \in I} \in \mathbf{Sch}^I$ with affine transition morphisms. Let $X = \varprojlim_{i \in I} X_i$ (X is still assumed to be noetherian and finite-dimensional). Then $\mathbf{DA}_c^{\text{coh}}(X)$ (resp. $\mathbf{DA}_{\text{hom},c}(X)$, $\mathbf{DA}_c^n(X)$, $\mathbf{DA}_{n,c}(X)$, $\mathbf{DA}_{n,c}^{\text{eff}}(X)$) is equal to the 2-colimit of the $\mathbf{DA}_c^{\text{coh}}(X_i)$ (resp. $\mathbf{DA}_{\text{hom},c}(X_i)$, $\mathbf{DA}_c^n(X_i)$, $\mathbf{DA}_{n,c}(X_i)$, $\mathbf{DA}_{n,c}^{\text{eff}}(X_i)$) via the pullback functors $(X \rightarrow X_i)^*$.*

Proof. Using the continuity result for morphisms in \mathbf{DA} from [Ayo14a, Proposition 3.19] and the arguments from [Ayo15, Corollaire 1.A.3] (using the description of compact objects discussed in Lemma 1.8), it is enough to prove the following lemma (which extends [Ayo15, Lemme 1.A.2]).

Lemma 1.22. *With the notation of the proposition, let Y be an X -scheme of finite presentation. Then there exists an $i \in I$ and an X_i -scheme Y_i of finite presentation such that $Y \simeq Y_i \times_{X_i} X$. Moreover, if Y/X is smooth (resp. of relative dimension $\leq n$, smooth of relative dimension $\leq n$), then Y_i can be chosen smooth (resp. of relative dimension $\leq n$, smooth of relative dimension $\leq n$).*

Proof. The first part is well known (see [Gro66, §8]). For the second part, the arguments of the proof of [Ayo15, Lemme 1.A.2] cover the case of smooth and smooth of relative dimension $\leq n$. The case of morphisms of relative dimension $\leq n$ (without smoothness assumption) is treated in [Sta, <http://stacks.math.columbia.edu/tag/0123>].

□

We deduce a useful punctual characterization of compact n -motives:

Proposition 1.23. *Let S be a scheme and $M \in \mathbf{DA}_c(S)$. Then the following are equivalent.*

- (i) $M \in \mathbf{DA}_c^{\text{coh}}(S)$ (resp. $\mathbf{DA}_{\text{hom},c}(S)$, $\mathbf{DA}_c^n(S)$, $\mathbf{DA}_{n,c}(S)$).
- (ii) For all $s \in S$, we have $s^* M \in \mathbf{DA}_c^{\text{coh}}(s)$ (resp. $\mathbf{DA}_{\text{hom},c}(s)$, $\mathbf{DA}_c^n(s)$, $\mathbf{DA}_{n,c}(s)$).

Proof. The direction (i) \Leftarrow (ii) follows from the stability established above of all these subcategories by pullbacks. For the other direction, we can assume S reduced by Corollary 1.18 (ii). We then proceed by noetherian induction. The case of generic points is settled by the hypothesis, we then use Proposition 1.21 to spread-out the property to an open set. We conclude by using Corollary 1.18 (i) and the induction hypothesis. □

1.4. Over a field. Over a field, Verdier duality interacts well with our subcategories of \mathbf{DA} .

Lemma 1.24. *Let k be a field. Write $\mathbb{D}_k := \underline{\text{Hom}}(-, \mathbb{Q}_k) : \mathbf{DA}(k)^{\text{op}} \rightarrow \mathbf{DA}(k)$ for the Verdier duality functor. We have*

$$\mathbb{D}_k(\mathbf{DA}_{\text{hom},c}(k)) \subset \mathbf{DA}_c^{\text{coh}}(k)$$

and \mathbb{D}_k restricts to anti-equivalences of categories

$$\begin{aligned} \mathbb{D}_k : \mathbf{DA}_{\text{hom},c}^{\text{gsm}}(k) &\xrightarrow{\sim} \mathbf{DA}_{\text{gsm},c}^{\text{coh}}(k) \text{ and} \\ \mathbb{D}_k : \mathbf{DA}_{n,c}^{\text{gsm}}(k) &\xrightarrow{\sim} \mathbf{DA}_{\text{gsm},c}^n(k). \end{aligned}$$

Proof. For X a separated scheme of finite type over k , consider the more general Verdier duality functor $\mathbb{D}_{X/k} := \underline{\text{Hom}}(-, \pi_X^! \mathbb{Q}_k) : \mathbf{DA}(X)^{\text{op}} \rightarrow \mathbf{DA}(X)$. By [Ayo14a, Théorèmes 8.12-8.14], this functor preserves compact objects and its restriction to $\mathbf{DA}_c(X)$ is an anti-autoequivalence which is its own quasi-inverse.

The behaviour of $\mathbb{D}_{X/k}$ with respect to the four operations is explained in [Ayo07a, Théorème 2.3.75]: informally, Verdier duality exchanges f_* and $f_!$, and f^* and $f^!$. Moreover, recall that,

for f smooth, relative purity provides an isomorphism $f_{\sharp}f^* \simeq f_!f^!$. This allows to compute the action of $\mathbb{D}_{X/k}$ on generating families. For instance, we have, for any f smooth, $\mathbb{D}_k(f_{\sharp}f^*\mathbb{Q}_X) \simeq \mathbb{D}_k(f_!f^!\mathbb{Q}_k) \simeq f_*f^*\mathbb{D}_k(\mathbb{Q}_k) \simeq f_*f^*\mathbb{Q}_k$, which is in $\mathbf{DA}^{\text{coh}}(k)$ by Proposition 1.11 (ii). This proves the first inclusion. For the equalities for geometrically smooth subcategories, note that if f is smooth projective (resp. smooth projective of relative dimension $\leq n$), the same computation shows that $\mathbb{D}_k(f_{\sharp}f^*\mathbb{Q}_X)$ is in $\mathbf{DA}_{\text{gsm}}^{\text{coh}}(k)$ (resp. $\mathbf{DA}_{\text{gsm}}^n(k)$). This proves one inclusion of the equalities, and the other follows by involutivity of \mathbb{D} . \square

As a consequence, when the base is the spectrum of a field, several of the notions we have introduced coincide.

Proposition 1.25. *Let k be any field; we have the following equalities.*

$$\begin{aligned} \mathbf{DA}_{\text{hom}}(k) &= \mathbf{DA}_{\text{hom}}^{\text{sm}}(k) = \mathbf{DA}_{\text{hom}}^{\text{gsm}}(k). \\ \mathbf{DA}^{\text{coh}}(k) &= \mathbf{DA}_{\text{sm}}^{\text{coh}}(k) = \mathbf{DA}_{\text{gsm}}^{\text{coh}}(k). \\ \mathbf{DA}_n(k) &= \mathbf{DA}_n^{\text{sm}}(k) = \mathbf{DA}_n^{\text{gsm}}(k). \\ \mathbf{DA}^n(k) &= \mathbf{DA}_{\text{sm}}^n(k) = \mathbf{DA}_{\text{gsm}}^n(k). \end{aligned}$$

The same equalities hold for the subcategories of compact objects, and \mathbb{D}_k restricts to anti-equivalences of categories:

$$\begin{aligned} \mathbb{D}_k : \mathbf{DA}_{\text{hom},c}(k) &\xrightarrow{\sim} \mathbf{DA}_c^{\text{coh}}(k) : \mathbb{D}_k \\ \mathbb{D}_k : \mathbf{DA}_{n,c}(k) &\xrightarrow{\sim} \mathbf{DA}_c^n(k) : \mathbb{D}_k \end{aligned}$$

Proof. In each case, we prove equality by showing that the generating family on each side lies in the other. The generating families used in the definitions of these categories are formed of compact objects, hence it suffices to prove the equalities for the subcategories of compact objects. By Lemma 1.5, we need only prove the inclusions

$$\begin{aligned} \mathbf{DA}_{\text{hom},c}(k) &\subset \mathbf{DA}_{\text{hom},c}^{\text{gsm}}(k), \\ \mathbf{DA}_c^{\text{coh}}(k) &\subset \mathbf{DA}_{\text{gsm},c}^{\text{coh}}(k), \\ \mathbf{DA}_{n,c}(k) &\subset \mathbf{DA}_{n,c}^{\text{gsm}}(k) \text{ and} \\ \mathbf{DA}_c^n(k) &\subset \mathbf{DA}_{\text{gsm},c}^n(k). \end{aligned}$$

The key is to prove the following claim: for all $n \in \mathbb{N}$, we have $\mathbb{D}_k(\mathbf{DA}_c^n(k)) \subset \mathbf{DA}_{n,c}^{\text{gsm}}(k)$.

Indeed, assume this claim for the next three paragraphs. Then by looking at generators we also get $\mathbb{D}_k(\mathbf{DA}_c^{\text{coh}}(k)) \subset \mathbf{DA}_{\text{hom},c}^{\text{gsm}}(k)$. By applying \mathbb{D}_k again and the equivalence of categories of Lemma 1.24, we get inclusions $\mathbf{DA}_c^n(k) \subset \mathbf{DA}_{\text{gsm},c}^n(k)$ and $\mathbf{DA}_c^{\text{coh}}(k) \subset \mathbf{DA}_{\text{gsm},c}^{\text{coh}}(k)$. By applying \mathbb{D}_k to the inclusion $\mathbb{D}_k(\mathbf{DA}_{\text{hom},c}(k)) \subset \mathbf{DA}_c^{\text{coh}}(k)$ of Lemma 1.24, we also obtain $\mathbf{DA}_{\text{hom},c}(k) \subset \mathbf{DA}_{\text{gsm},c}^{\text{coh}}(k)$. It remains to see that $\mathbf{DA}_c^n(k) \subset \mathbf{DA}_{\text{gsm},c}^n(k)$, which is slightly less clear.

Let $f : X \rightarrow k$ smooth of relative dimension $i \leq n$ (we can reduce to this case by considering connected components of X). By relative purity, we have $f_{\sharp}\mathbb{Q}_X(-n) \simeq f_!\mathbb{Q}_X(i-n)[2i]$ which is in $\mathbf{DA}_c^n(k)$ by Proposition 1.16 and 1.10.. This shows that $\mathbf{DA}_{n,c}(k)(-n) \subset \mathbf{DA}_c^n(k) = \mathbf{DA}_{\text{gsm},c}^n(k)$ (the last equality having just been established in the previous paragraph). Applying Verdier duality, we get $\mathbb{D}_k(\mathbf{DA}_{n,c}(k))(n) \subset \mathbb{D}_k(\mathbf{DA}_{\text{gsm},c}^n(k)) = \mathbf{DA}_{n,c}^{\text{gsm}}(k)$.

Another application of relative purity shows that $\mathbf{DA}_{n,c}^{\text{gsm}}(k)(-n) = \mathbf{DA}_{\text{gsm},c}^n(k)$. Putting everything together, we have $\mathbb{D}_k(\mathbf{DA}_{n,c}(k)) \subset \mathbf{DA}_{\text{gsm},c}^n(k) = \mathbb{D}_k(\mathbf{DA}_{n,c}^{\text{gsm}}(k))$ so by involutivity of \mathbb{D} we get the missing inclusion $\mathbf{DA}_c^n(k) \subset \mathbf{DA}_{\text{gsm},c}^n(k)$. This finishes the proof of the proposition modulo the claim.

Let us show that $\mathbb{D}_k(\mathbf{DA}_c^n(k)) \subset \mathbf{DA}_{n,c}^{\text{gsm}}(k)$. For simplicity, in the rest of the proof, we write $\pi_Y : Y \rightarrow k$ for the structural morphism of any k -scheme Y . Using the generating families, we reformulate the claim as follows: for $\pi_X : X \rightarrow k$ proper of relative dimension $\leq n$, we have $\mathbb{D}_k(\pi_{X*}\mathbb{Q}_X) \simeq \pi_{X!}\pi_X^!\mathbb{Q}_k$ in $\mathbf{DA}_{n,c}^{\text{gsm}}(k)$. Let $i : X_{\text{red}} \rightarrow X$. Then by localisation we have $\pi_{X!}\pi_X^!\mathbb{Q}_k \simeq \pi_{X!}i_!i^!\pi_X^!\mathbb{Q}_k \simeq \pi_{X_{\text{red}}!}\pi_{X_{\text{red}}}^!\mathbb{Q}_k$. Consequently, we can assume that X is reduced.

We first treat the case of a perfect field k . We proceed by induction on the dimension of X . When X is 0-dimensional, we see that π_X is finite étale because k is perfect and X is reduced, so that $\pi_{X!}\pi_X^!\mathbb{Q}_k \simeq \pi_{X\sharp}\pi_X^*\mathbb{Q}_k$ and we are done. For the induction step, we apply De Jong's resolution of singularities by alterations [dJ96, Theorem 4.1 + following remark]. We obtain an alteration

$h : \tilde{X} \rightarrow X$ with \tilde{X}/k a smooth projective variety. Recall that h is proper surjective and generically finite. We choose a diagram of schemes with cartesian squares

$$\begin{array}{ccccc} V & \xrightarrow{\tilde{j}} & \tilde{X} & \xleftarrow{\tilde{i}} & Z \\ \downarrow h_U & & \downarrow h & & \downarrow h_T \\ U & \xrightarrow{j} & X & \xleftarrow{i} & T \end{array}$$

with the following properties.

- T is a nowhere dense closed subset of X and U is its open complement.
- h_U can be written as the composite of a purely inseparable finite morphism followed by a finite étale morphism.

Starting from the distinguished colocalisation triangle for the pair (X, U) and applying $\pi_{X!}$, we obtain a triangle

$$\pi_{X!} i_* i^! \pi_X^! \mathbb{Q}_k \rightarrow \pi_{X!} \pi_X^! \mathbb{Q}_k \rightarrow (\pi_X)_! j_* j^! \pi_X^! \mathbb{Q}_k \xrightarrow{+}$$

that we rewrite as

$$(\pi_T)_! \pi_T^! \mathbb{Q}_k \rightarrow \pi_{X!} \pi_X^! \mathbb{Q}_k \rightarrow \pi_{X!} j_* j^! \pi_U^! \mathbb{Q}_k \xrightarrow{+}.$$

The left-hand term is in $\mathbf{DA}_{n,c}^{\text{gsm}}(k)$ by induction. To prove that the middle term is in $\mathbf{DA}_{n,c}^{\text{gsm}}(k)$, it remains to prove the same for the right-hand term. Since h_U is finite and the composite of a purely inseparable morphism followed by an étale morphism, the separation property of \mathbf{DA} [Ayo14a, Theorem 3.9] together with [Ayo07a, Corollaire 2.1.164] implies that there is a natural isomorphism of functors:

$$(h_U)_! h_U^! \simeq (h_U)_* h_U^*$$

Now, [Ayo07a, Lemma 2.1.165] implies that $\pi_U^! \mathbb{Q}_k$ is a direct factor of $(h_U)_* h_U^* \pi_X^! \mathbb{Q}_k$. Applying the isomorphism just above, we conclude that $\pi_U^! \mathbb{Q}_k$ is a direct factor of $(h_U)_! h_U^! \pi_U^! \mathbb{Q}_k$. This last motive is isomorphic to $(h_U)_* \pi_V^! \mathbb{Q}_k \simeq (h_U)_* \tilde{j}^* \pi_{\tilde{X}}^! \mathbb{Q}_k$ because h_U is proper and \tilde{j} is étale. We get that $\pi_{X!} j_* \pi_U^! \mathbb{Q}_k$ is a direct factor of $\pi_{X!} j_* (h_U)_* \tilde{j}^* \pi_{\tilde{X}}^! \mathbb{Q}_k \simeq \pi_{\tilde{X}!} \tilde{j}_* \tilde{j}^* \pi_{\tilde{X}}^! \mathbb{Q}_k$. Applying localisation to the pair (\tilde{X}, V) , the fact that \tilde{X}/k is smooth projective and the induction hypothesis for Z shows that this last object is in $\mathbf{DA}_{n,c}^{\text{gsm}}(k)$. This concludes the proof when k is perfect.

We now treat the case of a general field k . By the perfect field case and continuity for $\mathbf{DA}_{n,c}^{\text{gsm}}(-)$ (Proposition 1.21) applied to the spectrum of the perfect closure of k , we see that there exists a finite purely inseparable extension l/k with $(l/k)_* \pi_{X!} \pi_X^! \mathbb{Q}_k$ in $\mathbf{DA}_{n,c}^{\text{gsm}}(l)$. By the separation property, we have an isomorphism of functors $\text{id} \simeq (l/k)_* (l/k)^*$, so that it is enough to show Lemma 1.26 below. This completes the proof of the claim, hence of the chains of equalities in the proposition.

Finally, the Verdier duality statement is just a restatement of Lemma 1.24 in the light of these chains of equalities. \square

Lemma 1.26. *For a finite field extension l/k and $g : Y \rightarrow \mathbf{Spec}(l)$ a smooth projective morphism of relative dimension $\leq n$, there exists a smooth projective variety $g' : Y' \rightarrow k$ of dimension $\leq n$ such that $(l/k)_* g_{\#} \mathbb{Q}_Y \simeq g'_{\#} \mathbb{Q}_{Y'} \in \mathbf{DA}_{n,c}^{\text{gsm}}(k)$.*

Proof. We immediately reduce to the case of l/k purely inseparable. By treating separately the connected components of Y , we can assume that Y is of dimension n . Let $F : \mathbf{Spec}(l) \rightarrow \mathbf{Spec}(l)$ be an high enough power of the Frobenius of l that factors through k . We denote again by F the induced morphism $\mathbf{Spec}(k) \rightarrow \mathbf{Spec}(l)$ and its natural lift $\mathbf{Spec}(k) \rightarrow \mathbf{Spec}(k)$ (the corresponding power of Fr_k). We have the following diagram of schemes, where the upper square is cartesian:

$$\begin{array}{ccc} Y' & \xrightarrow{F_Y} & Y \\ \pi_{Y'} \downarrow & & \downarrow \pi_Y \\ \mathbf{Spec}(k) & \xrightarrow{F} & \mathbf{Spec}(l) \\ & \searrow F & \downarrow (l/k) \\ & & \mathbf{Spec}(k). \end{array}$$

By base change, the k -scheme Y' is smooth projective and the morphism F_Y is finite purely inseparable. By the separation property of \mathbf{DA} , we have

$$(l/k)_*(\pi_Y)_*\mathbb{Q}_Y \simeq (l/k)_*(\pi_Y)_*(F_Y)_*\mathbb{Q}_{Y'} \simeq (l/k)_*F_*(\pi_{Y'})_*\mathbb{Q}_{Y'} \simeq F_*(\pi_{Y'})_*\mathbb{Q}_{Y'}.$$

Let $\mathrm{Fr}_{Y'}$ be the corresponding power of the absolute Frobenius on Y' . By naturality of the absolute Frobenius, we have $\pi_{Y'} \circ \mathrm{Fr}_{Y'} = F \circ \pi_{Y'} : Y' \rightarrow \mathbf{Spec}(k)$. We deduce that

$$F_*(\pi_{Y'})_*\mathbb{Q}_{Y'} \simeq (\pi_{Y'})_*(\mathrm{Fr}_{Y'})_*\mathbb{Q}_{Y'} \simeq (\pi_{Y'})_*\mathbb{Q}_{Y'} \in \mathbf{DA}_{\mathrm{gsm}}^n(k),$$

where the last isomorphism follows by separation. By relative purity and the projection formula, we deduce that

$$\begin{aligned} (l/k)_*(\pi_Y)_\# \mathbb{Q}_Y &\simeq (l/k)_*((\pi_Y)_*\mathbb{Q}_Y \otimes \mathbb{Q}_l(n)[2n]) \\ &\simeq (l/k)_*((\pi_Y)_*\mathbb{Q}_Y) \otimes \mathbb{Q}_k(n)[2n] \\ &\simeq (\pi_{Y'})_*\mathbb{Q}_{Y'} \otimes \mathbb{Q}_k(n)[2n] \\ &\simeq (\pi_{Y'})_\# \mathbb{Q}_{Y'}. \end{aligned}$$

This completes the proof of the lemma. \square

1.5. Homological vs cohomological motives.

Proposition 1.27. *Let S be a scheme, $n \geq 0$. We have*

$$\mathbf{DA}_{(c)}^n(S) = \mathbf{DA}_{n,(c)}(S)(-n)$$

In particular, we have $\mathbf{DA}_{(c)}^0(S) = \mathbf{DA}_{0,(c)}(S)$.

Proof. In both directions, it is enough to check the inclusion for a family of compact generators.

Let $f : X \rightarrow S$ be a smooth morphism of relative dimension $i \leq n$ (we can reduce to this case by considering connected components of S and X). By relative purity, we have

$$f_\# \mathbb{Q}_X(-n) \simeq f! \mathbb{Q}_X(i-n)[2i]$$

which is in $\mathbf{DA}^n(S)$ by Propositions 1.16 and 1.10.

The other inclusion is true for smooth cohomological n -motives by the same relative purity argument. For general compact cohomological n -motives (which include the generating family), we argue as follows. By Corollary 1.18 (ii), we can assume S reduced. We then proceed by noetherian induction. Let $M \in \mathbf{DA}^n(S)$. The restriction of M to any generic point of S is smooth by Proposition 1.25. There we can apply the smooth case and see that $\eta^* M \in \mathbf{DA}_{n,c}(\eta)(-n)$ for any generic point η of S . Then we apply continuity for compact homological n -motives (Proposition 1.21) to find a dense open immersion $j : U \rightarrow S$ with $j^* M \in \mathbf{DA}_{n,c}(U)(-n)$. Applying the induction hypothesis, localisation and the fact that i_* preserves homological n -motives for i closed immersion (Proposition 1.17 (ii)) completes the proof. \square

1.6. Nearby cycles. To conclude this section, we prove a result about the nearby cycles functor and n -motives.

Let R be an excellent henselian discrete valuation ring and let $S = \mathbf{Spec}(R)$, η be the generic fiber and σ be the closed fiber. Fix a separable closure K^{sep} of $K = \mathrm{Frac}(R)$ and let $\bar{\sigma}$ be the spectrum of its residue field. Let X be an S -scheme and $f : X \rightarrow \mathbb{A}_S^1$ a morphism. There is a tame nearby motive functor $\Psi_f^{\mathrm{mod}} : \mathbf{DA}(X_\eta) \rightarrow \mathbf{DA}(X_\sigma)$ and a nearby motive functor $\Psi_f : \mathbf{DA}(X_\eta) \rightarrow \mathbf{DA}(X_{\bar{\sigma}})$ (see [Ayo14a, Section 10, Définition 10.14]), which are part of specialisation systems in the sense of [Ayo07b, Definition 3.1.1]. An important case is when $f = \pi$ is a uniformizer of R .

Lemma 1.28. *With the above notations, the functors Ψ_f^{mod} and Ψ_f commute with infinite sums.*

Proof. The fact that Ψ_f^{mod} commutes with infinite sums follows from [Ayo07b, Lemma 3.2.10], which applies because of the definition of the specialisation system Ψ^{mod} as $\mathcal{R} \bullet \xi$ with \mathcal{R} a certain diagram of schemes and $\xi = i^* j_*$ the canonical specialisation system, which itself commutes with infinite sums [Ayo07b, Definition 3.5.6]. If the residual characteristic of R is 0, then Ψ_f is simply defined as the composition $(X_{\bar{\sigma}}/X_\sigma)^* \Psi_f^{\mathrm{mod}}$ and we are done. Let us assume that the residual characteristic is $p > 0$.

Recall that the functor Ψ_f of [Ayo14a, Définition 10.14] is then constructed from Ψ_f^{mod} via an homotopy colimit along all the finite extensions of K contained in a fixed maximum p -primary extension inside K^{sep} of the maximal unramified extension K^{nr} (such extensions exist by the theorem of Schur-Zassenhaus).

We make this second step slightly more explicit, as suggested in [Ayo14a, Remark 10.15]. If M_δ/K^{nr} is such a fixed maximum p -primary extension, let \mathcal{L} be the poset of all the finite subextensions $K \subset L \subset M_\delta$ ordered by the reverse of inclusion. Then there is a diagram of schemes (T_L, \mathcal{L}) where T_L is the normalisation of S inside L/K , along with diagrams (η_L, \mathcal{L}) and (σ_L, \mathcal{L}) of generic and special fibers. We have a morphism $\gamma : (T_L, \mathcal{L}) \rightarrow S$ and we pullback the diagram of schemes over S used to compute Ψ_f^{mod} along this morphism. We also use the notation $\tilde{\mathcal{L}} = \mathcal{L} \times \Delta \times (\mathbb{N}')^\times$ where Δ is the simplicial category and $(\mathbb{N}')^\times = \{n \in \mathbb{N}^\times | \text{char}(\sigma) \mid n\}$. Altogether, we get a commutative diagram of diagrams of schemes with cartesian squares:

$$\begin{array}{ccccccc} (\mathcal{R}'_{f_L}, \tilde{\mathcal{L}}) & \xrightarrow{\theta_{f_L}^{\mathcal{R}'}} & (X_{\eta_L}, \tilde{\mathcal{L}}) & \xrightarrow{j} & (X_{T_L}, \tilde{\mathcal{L}}) & \xleftarrow{i} & (X_{\sigma_L}, \tilde{\mathcal{L}}) \xrightarrow{p_{\Delta \times (\mathbb{N}')^\times}} (X_{\sigma_L}, \mathcal{L}) \\ \downarrow f_{\eta_L} & & \downarrow f_{\eta_L} & & \downarrow f_{\mathcal{L}} & & \downarrow f_{\sigma_L} \\ (\mathcal{R}'_{T_L}, \tilde{\mathcal{L}}) & \xrightarrow{\theta_{T_L}^{\mathcal{R}'}} & ((\mathbb{G}_m)_{T_L}, \tilde{\mathcal{L}}) & \xrightarrow{j} & (\mathbb{A}_{T_L}^1, \tilde{\mathcal{L}}) & \xleftarrow{i} & (T_L, \tilde{\mathcal{L}}) \xrightarrow{p_{\Delta \times (\mathbb{N}')^\times}} (T_L, \mathcal{L}) \end{array}$$

We also have morphisms $\bar{\pi} : (X_{\bar{\sigma}}, \mathcal{L}) \rightarrow (X_{\sigma_L}, \mathcal{L})$, $p_{\mathcal{L}} : (X_{\bar{\sigma}}, \mathcal{L}) \rightarrow X_{\bar{\sigma}}$ and $\gamma_{X_\eta} : (X_{\eta_L}, \mathcal{L}) \rightarrow X_\eta$. We can finally define:

$$\Psi_f := (p_{\mathcal{L}})_\# \bar{\pi}^* (p_{\Delta \times (\mathbb{N}')^\times})_\# i^* j_* (\theta_{f_L}^{\mathcal{R}'})_* (\theta_{f_L}^{\mathcal{R}'})^* (p_{\Delta \times (\mathbb{N}')^\times})^* \gamma_{X_\eta}^*$$

To prove that Ψ_f commutes with infinite sums from this formula, we adapt the proof of [Ayo07b, Lemma 3.2.10]. The functors $(p_{\mathcal{L}})_\#$, $\bar{\pi}^*$, $(p_{\Delta \times (\mathbb{N}')^\times})_\#$, i^* , $(\theta_{f_L}^{\mathcal{R}'})^*$, $(p_{\Delta \times (\mathbb{N}')^\times})^*$ and $\gamma_{X_\eta}^*$ are all left adjoints, hence they commute with small sums. To show that j_* and $(\theta_{f_L}^{\mathcal{R}'})_*$ commute with small sums, we use the fact that the corresponding morphisms of diagrams of schemes are “of geometric type”, i.e. the underlying morphism of small categories is an isomorphism. This implies that the commutation with small sums can be checked after restriction to a vertex in the corresponding diagram (because $\mathbf{DA}(-)$ is an algebraic derivator, the family of such functors for a given diagram of schemes is conservative), and then we find normal pushforward functors in $\mathbf{DA}(-)$ which do commute with small sums. This concludes the proof. \square

Proposition 1.29. *With the notations above, the functor Ψ_π^{mod} (resp. Ψ_π):*

- (1) *sends $\mathbf{DA}^{\text{coh}}(\eta)$ to $\mathbf{DA}^{\text{coh}}(\sigma)$ (resp. to $\mathbf{DA}^{\text{coh}}(\bar{\sigma})$),*
- (2) *sends $\mathbf{DA}^n(\eta)$ to $\mathbf{DA}^n(\sigma)$ (resp. to $\mathbf{DA}^n(\bar{\sigma})$) for any $n \geq 0$,*
- (3) *sends $\mathbf{DA}_{\text{hom}}(\eta)$ to $\mathbf{DA}_{\text{hom}}(\sigma)$ (resp. to $\mathbf{DA}_{\text{hom}}(\bar{\sigma})$), and*
- (4) *sends $\mathbf{DA}_n(\eta)$ to $\mathbf{DA}_n(\sigma)$ (resp. to $\mathbf{DA}_n(\bar{\sigma})$).*

Similar results hold for the subcategories of compact objects.

Proof. By Lemma 1.28, the functors Ψ_π^{mod} and Ψ_π commute with small sums. This shows that we only need to establish the results for compact objects. Using the fact that nearby cycles commute with Verdier duality on constructible objects [Ayo14a, Théorème 10.20] together with the fact that Verdier duality exchanges $\mathbf{DA}_c^n(k)$ and $\mathbf{DA}_{n,c}(k)$ (Lemma 1.24) allows us to deduce (iii) and (iv) from (i) and (ii).

It remains to show the property (i) (resp. (ii)) for the compact generators of $\mathbf{DA}^{\text{coh}}(\eta)$ of the form $f_* \mathbb{Q}_X$ with $f : X \rightarrow \eta$ proper (resp. of $\mathbf{DA}^n(\eta)$ of the form $f_* \mathbb{Q}_X$ with $f : X \rightarrow \eta$ a proper morphism of relative dimension $\leq n$). Using Mayer-Vietoris for closed covers, we can further assume that X is irreducible. Moreover, since these objects are constructible, the fact that Ψ_π is computed as Ψ_π^{mod} after a large enough finite extension [Ayo14a, Théorème 10.13, Remarque 10.15] reduces the proof to the case of Ψ_π^{mod} .

Let $f^0 : X^0 \rightarrow \eta$ be proper of dimension $\leq n$. We show that $\Psi_\pi^{\text{mod}} f_*^0 \mathbb{Q}_{X^0}$ is in $\mathbf{DA}^n(\sigma)$ by induction on n : this is enough to prove both (i) and (ii). In the case $n = 0$, we reduce by localisation to $X^0 = \eta$ and the result is immediate from $\Psi_{\text{id}}^{\text{mod}} \mathbb{Q}_\eta \simeq \mathbb{Q}_\sigma$.

In general, choose a proper flat morphism $f : X \rightarrow S$ such that $X^0 = X_\eta$ irreducible and $f^0 = f_\eta$.

The special fiber X_σ is also of relative dimension $\leq n$. Because Ψ^{mod} is a specialisation system [Ayo07b, Definition 3.1.1] and f is proper, we have an isomorphism

$$\beta : \Psi_\pi^{\text{mod}}(f_\eta)_* \mathbb{Q}_{X_\eta} \xrightarrow{\sim} (f_\sigma)_* \Psi_f^{\text{mod}} \mathbb{Q}_{X_\eta}.$$

To simplify the notation, for any S -scheme $g : W \rightarrow S$, we write

$$\Psi_W := (g_\sigma)_* \Psi_f^{\text{mod}} \mathbb{Q}_{X_\eta}.$$

Now we want to reduce to a situation with a better behaved special fiber. We apply De Jong's theorem on semi-stable reduction by alterations [dJ96, Theorem 4.5]. There exists an henselian discrete valuation ring \tilde{S} finite over S and a commutative square

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{p} & X \\ g \downarrow & & \downarrow f \\ \tilde{S} & \xrightarrow{\pi} & S \end{array}$$

such that

- \tilde{X} is regular and strictly semi-stable over \tilde{S} (in the sense of [dJ96, 2.16]), and
- p is an alteration.

Let V be an open set contained in \tilde{X}_η such that p_V is the composition of a finite flat purely inseparable morphism followed by a finite étale morphism, and consider $Z = \tilde{X} \setminus V$ with its reduced scheme structure. We have a commutative diagram (not necessarily cartesian, but with $U := p(V)$ open by flatness and $T := p(Z) = X \setminus U$ by surjectivity)

$$\begin{array}{ccccc} V & \xrightarrow{\tilde{j}} & \tilde{X} & \xleftarrow{\tilde{i}} & Z \\ p_U \downarrow & & p \downarrow & & p_Z \downarrow \\ U & \xrightarrow{j} & X & \xleftarrow{i} & T \end{array}$$

We have distinguished triangles in $\mathbf{DA}(X_\eta)$,

$$(j_\eta)_! \mathbb{Q}_{U_\eta} \rightarrow \mathbb{Q}_{X_\eta} \rightarrow (i_\eta)_* \mathbb{Q}_{T_\eta} \xrightarrow{+}$$

and

$$(p_{U_\eta})_*(\tilde{j}_\eta)_! \mathbb{Q}_{V_\eta} \rightarrow (p_\eta)_* \mathbb{Q}_{X'_\eta} \rightarrow (i_\eta)_*(p_{Z_\eta})_* \mathbb{Q}_{Z_\eta} \xrightarrow{+}.$$

After applying Ψ_f^{mod} , pushing forward to σ (we forget temporarily the \tilde{S} -scheme structure of X' , V and Z) and using properness of p and i , we get distinguished triangles

$$(f_\sigma)_* \Psi_f^{\text{mod}}(j_\eta)_! \mathbb{Q}_{U_\eta} \rightarrow \Psi_X \rightarrow \Psi_T \xrightarrow{+}$$

and

$$(f_\sigma)_* \Psi_f^{\text{mod}}(p_{U_\eta})_*(\tilde{j}_\eta)_! V_\eta \rightarrow \Psi_{\tilde{X}} \rightarrow \Psi_Z \xrightarrow{+}$$

in $\mathbf{DA}(\sigma)$.

Since $\dim(T_\eta), \dim(Z_\eta) < n$, the terms Ψ_T and Ψ_Z are in $\mathbf{DA}^n(\sigma)$ by the induction hypothesis. On the other hand, since p_V is the composition of a finite purely inseparable morphism and a finite étale morphism, the separation property of $\mathbf{DA}(-)$ together with [Ayo07a, Lemme 2.1.165] imply that $(j_\eta)_! \mathbb{Q}_{U_\eta}$ is a direct factor of $(p_{U_\eta})_*(\tilde{j}_\eta)_! V_\eta$. It is thus enough to prove that $\Psi_{\tilde{X}}$ is in $\mathbf{DA}^n(\sigma)$.

We will prove that $(g_\sigma)_* \Psi_g^{\text{mod}} \mathbb{Q}_{X'_\eta} \in \mathbf{DA}^n(\tilde{S})$. Since $\pi : \tilde{S} \rightarrow S$ is finite, this will imply the same for $\Psi_{\tilde{X}}$ by Proposition 1.16 (ii).

Write $\tilde{X}_s = \cup_{k=1}^m D_k$ as a union of its irreducible components. For each $I \subset [1, m]$ write $D_I = \bigcap_{k \in I} D_k$ for the scheme-theoretic intersection. For $I \subset J$, write $(i_J^I) : D_J \rightarrow D_I$ for the corresponding closed immersion. For any I , write

$$D_I^\circ := D_I \setminus \bigcup_{l \notin I} D_l$$

and $j_I : D_I^\circ \rightarrow D_I$ for the corresponding open immersion. By [Ayo07a, Lemme 2.2.31], it is enough to prove that for any $I \neq \emptyset$, we have

$$(f_\sigma)_*(i_I)_* i_I^! \Psi_g^{\text{mod}} \mathbb{Q}_{\tilde{X}_\eta} \in \mathbf{DA}^n(\sigma).$$

Let I be such an index set, and let $k \in I$. By purity for the regular codimension 1 closed immersion $i_k^!$ and [Ayo07b, Theorem 3.3.43] (which applies to $\mathbf{DA}^{\text{ét}}(-, \mathbb{Q})$), we have

$$\begin{aligned} i_I^! \Psi_g \mathbb{Q}_{\tilde{X}_\eta} &\simeq (i_I^k)^! i_k^! \Psi_g \mathbb{Q}_{\tilde{X}_\eta} \\ &\simeq (i_I^k)^* i_k^* \Psi_g \mathbb{Q}_{\tilde{X}_\eta}(1)[2] \\ &\simeq (i_I^k)^* (j_k)_* (j_k)^* i_k^* \Psi_g \mathbb{Q}_{\tilde{X}_\eta}(1)[2]. \end{aligned}$$

Because D_k is smooth over σ by semi-stability, we have $(j_k)^*(i_k)^! \Psi_g \mathbb{Q}_{\tilde{X}_\eta} \simeq (i_k \circ j_k)^! \Psi_g \mathbb{Q}_{\tilde{X}_\eta} \simeq (i_k \circ j_k)^* \Psi_g \mathbb{Q}_{\tilde{X}_\eta} \simeq \mathbb{Q}_{D_k^\circ}$ by axiom (SPE2).2 of specialization systems [Ayo07b, Definition 3.1.1] and [Ayo14a, Theorem 10.6] (with $e = 1$). So we are reduced to computing $(i_I^k)^*(j_k)_* \mathbb{Q}_{D_k^\circ}$. Relative purity, localisation and a further induction on branches imply that this motive is an iterated extension of sums of negative Tate twists $\mathbb{Q}_{D_I}(-d)[-2d]$ for $d \leq \text{card}(I \setminus \{k\})$, so it is in $\mathbf{DA}^{\text{card}(I)-1}(D_I)$ by Proposition 1.10 (iii). Since D_I is of relative dimension $\leq n - \text{card}(I)$ over σ , an application of Proposition 1.16 (ii) and the observation that

$$1 + (\text{card}(I) - 1) + (n - \text{card}(I)) = n$$

finishes the proof. \square

2. COMMUTATIVE GROUP SCHEMES AND MOTIVES

Several motives of interest for this paper are obtained from group schemes or complexes of group schemes. The main examples we are interested in are smooth commutative group schemes, Deligne 1-motives (Appendix A), and the smooth Picard complex (Section 2.3).

2.1. Motives of commutative group schemes. In this section, we introduce the relevant definitions and reformulate results from [AHPL14] and [Org04] in this language. For the rest of the section, fix a noetherian finite-dimensional scheme S .

In [AHPL14, Thm D.1], we constructed a functorial cofibrant resolution of the sheaf $G \otimes \mathbb{Q}$ for G a smooth (locally of finite type) commutative group scheme over S . Let us recall the statement.

Lemma 2.1. [AHPL14, Thm D.1] *Let (S, τ) be a Grothendieck site. We denote $\mathbb{Z}(-)$ the functor “free abelian group sheaf” (the sheafification of the sectionwise free abelian group functor).*

There is a functor:

$$A : \mathbf{Sh}_\tau(S, \mathbb{Z}) \rightarrow \mathbf{Cpl}_{\geq 0} \mathbf{Sh}_\tau(S, \mathbb{Z})$$

together with a natural transformation

$$r : A \rightarrow (-)[0]$$

satisfying the following properties.

- (1) *For all $\mathcal{G} \in \mathbf{Sh}_\tau(S, \mathbb{Z})$ and $i \geq 0$, the sheaf $A(\mathcal{G})_i$ is of the form $\bigoplus_{j=0}^{d(i)} \mathbb{Z}(\mathcal{G}^{a(i,j)})$ for some $d(i), a(i, j) \in \mathbb{N}$.*
- (2) *There is a natural transformation $\tilde{a} : \mathbb{Z}(-)[0] \rightarrow A$ which lifts the addition map $a : \mathbb{Z}(-) \rightarrow \text{id}$; that is, one has $a[0] = r\tilde{a}$.*
- (3) *The functor A and the transformations r and \tilde{a} are compatible with pullbacks by morphisms of sites.*
- (4) *The map $r \otimes \mathbb{Q}$ is a quasi-isomorphism.*

Let us make more explicit the statement in 3. Recall that we use underlines to denote underived functors between categories of complexes. For a morphism of sites $F : \mathcal{S}' \rightarrow \mathcal{S}$, and \mathcal{G} as in the theorem, we assert that there exists an isomorphism of complexes $b_{F, \mathcal{G}} : \underline{F}^*(A(\mathcal{G})) \rightarrow A(\underline{F}^*(\mathcal{G}))$

which is termwise compatible with the standard isomorphisms $\underline{F}^*\mathbb{Z}(\mathcal{G}^{a(i,j)}) \simeq \mathbb{Z}(\underline{F}^*\mathcal{G}^{a(i,j)})$ and which makes the diagram

$$\begin{array}{ccc} \underline{F}^*(A(\mathcal{G})) & \xrightarrow{\underline{F}^*(r(\mathcal{G}))} & \underline{F}^*\mathcal{G} \\ b_{F,\mathcal{G}} \downarrow & \nearrow r(\underline{F}^*\mathcal{G}) & \\ A(\underline{F}^*\mathcal{G}) & & \end{array}$$

commute.

Proposition 2.2. *Let K_* be a bounded complex of smooth commutative group schemes over S and $f : T \rightarrow S$ a morphism of schemes. We have a natural isomorphism*

$$R_f : f^*(K_* \otimes \mathbb{Q}) \xrightarrow{\sim} \underline{f}^*(K_* \otimes \mathbb{Q})$$

in $D(\mathbf{Sm}/S)$. Moreover, R_f is compatible with further pullbacks: for $g : U \rightarrow T$, the diagram

$$\begin{array}{ccc} g^*f^*(K_* \otimes \mathbb{Q}) & \xrightarrow{\sim} (fg)^*(K_* \otimes \mathbb{Q}) \xrightarrow{R_{fg}} \underline{(fg)}^*(K_* \otimes \mathbb{Q}) \\ R_f \downarrow \sim & & \downarrow \sim \\ g^*\underline{f}^*(K_* \otimes \mathbb{Q}) & \xrightarrow[\sim]{R_g} \underline{g}^*\underline{f}^*(K_* \otimes \mathbb{Q}) \end{array}$$

commutes.

Proof. We apply Lemma 2.1 to the individual sheaves K_n , and use the natural functoriality of the construction. This yields a double complex $A(K_*)_{i \in \mathbb{Z}, j \in \mathbb{N}}$ together with a map $r : A(K_*) \rightarrow K_*$. We then tensor by \mathbb{Q} and take the total complex along the second index. This yields a complex $B_{\mathbb{Q}}(K)_*$ of sheaves of \mathbb{Q} -vector spaces on $(\mathbf{Sm}/S)_{\text{ét}}$ together with a map $r_{\mathbb{Q}}(K_*) : B_{\mathbb{Q}}(K)_* \rightarrow K_*$ with the following properties.

- (i) For all $i \in \mathbb{Z}$, the sheaf $B_{\mathbb{Q}}(K)_i$ is of the form $\mathbb{Q}(H_i)$ for some smooth commutative group scheme H_i over S (a fiber product of various copies of the K_n 's); therefore, $B_{\mathbb{Q}}(K_*)$ is a projective object in $\mathbf{Cpl}(\mathbf{Sh}_{\text{ét}}(\mathbf{Sm}/S, \mathbb{Q}))$.
- (ii) The map $r_{\mathbb{Q}}(K_*)$ is a quasi-isomorphism, hence a projective resolution of $K_* \otimes \mathbb{Q}$.
- (iii) The formation of $B_{\mathbb{Q}}(K_*)$ and $r_{\mathbb{Q}}(K_*)$ is compatible with (underived) pullback, in the sense that, for any morphism $f : T \rightarrow S$, there exists an isomorphism of complexes $b_{f,K_*} : \underline{f}^*(B_{\mathbb{Q}}(K_*)) \rightarrow B_{\mathbb{Q}}(\underline{f}^*K_*)$ which makes the following diagram in $\mathbf{Cpl}(\mathbf{Sh}_{\text{ét}}(\mathbf{Sm}/T, \mathbb{Q}))$ commutes.

$$\begin{array}{ccc} \underline{f}^*(B_{\mathbb{Q}}(K_*)) & \xrightarrow{\underline{f}^*(r_{\mathbb{Q}}(K_*))} & \underline{f}^*(K_* \otimes \mathbb{Q}) \\ b_{f,K_*} \downarrow & \nearrow r(\underline{f}^*K_*) & \\ B_{\mathbb{Q}}(\underline{f}^*(K_*)) & & \end{array}$$

Because $r_{\mathbb{Q}}$ is a projective resolution, we have an isomorphism in $D(\mathbf{Sm}/S)$

$$f^*(K_* \otimes \mathbb{Q}) \xleftarrow[\sim]{f^*(r_{\mathbb{Q}}(K_*))} f^*(B_{\mathbb{Q}}(K_*)) \simeq \underline{f}^*(B_{\mathbb{Q}}(K_*)).$$

We define R_f as the composition

$$f^*(K_* \otimes \mathbb{Q}) \xrightarrow[\sim]{f^*(r_{\mathbb{Q}}(K_*))^{-1}} \underline{f}^*(B_{\mathbb{Q}}(K_*)) \xrightarrow[\sim]{\underline{f}^*(r_{\mathbb{Q}})} \underline{f}^*(K_* \otimes \mathbb{Q}).$$

It remains to check the compatibility with further pullbacks. Let $g : U \rightarrow T$ be a morphism of schemes. The reader is invited to contemplate the following diagram in $D(\mathbf{Sm}/S)$ (where the unlabelled maps are either cocycle isomorphisms for the pullbacks - derived and not - or isomorphisms

of the form $h^*(C) \simeq \underline{h}^*(C)$ for C cofibrant).

$$\begin{array}{ccccc}
g^* f^*(K_* \otimes \mathbb{Q}) & \xleftarrow[\sim]{g^* f^* r_{\mathbb{Q}}} & g^* f^* B_{\mathbb{Q}}(K_*) & \xleftarrow{\sim} & g^* \underline{f}^* B_{\mathbb{Q}}(K_*) & \xrightarrow{g^* \underline{f}^* r_{\mathbb{Q}}} & g^* \underline{f}^*(K_* \otimes \mathbb{Q}) \\
\downarrow \sim & & \uparrow & & \uparrow & & \uparrow g^* r_{\mathbb{Q}} \\
(fg)^*(K_* \otimes \mathbb{Q}) & \sim & & & & & g^* B_{\mathbb{Q}}(\underline{f}^*(K_*)) \\
\uparrow (fg)^* r_{\mathbb{Q}} & \nwarrow & \nearrow & & \nwarrow g^* b_{f, K_*} & & \uparrow \sim \\
(fg)^* B_{\mathbb{Q}}(K_*) & & & & & & g^* B_{\mathbb{Q}}(\underline{f}^* K_*) \\
\uparrow \sim & & \uparrow & & \xleftarrow{g^* b_{f, K_*}} & & \downarrow \sim \\
(\underline{fg})^* B_{\mathbb{Q}}(K_*) & \xleftarrow[\sim]{\sim} & \underline{g}^* \underline{f}^* B_{\mathbb{Q}}(K_*) & \xleftarrow{\sim} & \underline{g}^* B_{\mathbb{Q}}(\underline{f}^* K_*) & & \downarrow \underline{g}^* r_{\mathbb{Q}} \\
\downarrow (fg)^* r_{\mathbb{Q}} & & \downarrow & & \downarrow \underline{g}^* r_{\mathbb{Q}} & & \downarrow \underline{g}^* r_{\mathbb{Q}} \\
(\underline{fg})^*(K_* \otimes \mathbb{Q}) & \xleftarrow[\sim]{\sim} & & & \underline{g}^* \underline{f}^*(K_* \otimes \mathbb{Q}) & \xleftarrow[\sim]{\sim} & \underline{g}^* \underline{f}^*(K_* \otimes \mathbb{Q})
\end{array}$$

(A) (B) (C) (D) (E) (F) (G)

The quadrangles (A) and (F) commute because of the naturality of the cocycle isomorphisms for pullbacks. The triangle (B) and the quadrangle (E) commute trivially. The triangles (C) and (G) commute because of property (iii) above. Finally, the quadrangle (D) commutes because the cocycle isomorphisms for derived and underived pullbacks are compatible. \square

Corollary 2.3. *Let K_* be a bounded complex of smooth commutative group schemes over S and $f : T \rightarrow S$ be a morphism of schemes. We have natural isomorphisms*

$$R_f : f^* K_* \otimes \mathbb{Q} \xrightarrow{\sim} \underline{f}^*(K_* \otimes \mathbb{Q})$$

in $\mathbf{DA}^{\text{eff}}(S)$ and

$$R_f : f^* \Sigma^\infty(K_* \otimes \mathbb{Q}) \xrightarrow{\sim} \Sigma^\infty \underline{f}^*(K_* \otimes \mathbb{Q})$$

in $\mathbf{DA}(S)$. These isomorphisms are compatible with further pullbacks in the same way as in the previous proposition.

Proof. The first isomorphism follows directly from Proposition 2.2. The second follows from the first together with the commutation of f^* and Σ^∞ . \square

For some arguments, we need to use motives with transfers of commutative group schemes over a field. Let k be a field and G a smooth (locally of finite type) commutative group scheme over k . Recall that the étale sheaf G on \mathbf{Sm}/S admits a canonical structure of sheaf with transfers [BVK, Lemma 1.4.4], which is functorial in G . We write G^{tr} for the resulting sheaf with transfers. Recall that there are adjunctions

$$a_{\text{tr}} : \mathbf{DA}^{(\text{eff})}(k) \rightleftarrows \mathbf{DM}^{(\text{eff})}(k) : o^{\text{tr}}$$

which relate motives with and without transfers.

Proposition 2.4. [AHPL14, Proposition 2.10] *Let S be an excellent scheme and $M = G_{\mathbb{Q}}^{\text{tr}}$ with G a smooth commutative group scheme over S . Then the counit morphisms*

$$a_{\text{tr}} o^{\text{tr}} M \xrightarrow{\sim} M$$

in $\mathbf{DM}^{\text{eff}}(S)$ and

$$a_{\text{tr}} o^{\text{tr}} \Sigma_{\text{tr}}^\infty M \xrightarrow{\sim} \Sigma_{\text{tr}}^\infty M$$

in $\mathbf{DM}(S)$ are isomorphisms.

An important consequence for us is the following computation, which consists of translating a classical result of Voevodsky to our context, and which we will generalize later on.

Proposition 2.5. *Let k be a field and C/k be a smooth projective geometrically connected curve. There exists a direct sum decomposition*

$$M(C) \simeq \mathbb{Q} \oplus \Sigma^\infty \text{Jac}(C)_{\mathbb{Q}} \oplus \mathbb{Q}(1)[2]$$

in $\mathbf{DA}(k)$.

Proof. We first assume k perfect. For a smooth projective connected curve C over k with a rational point, Voevodsky has computed the motive $M_{\text{tr}}^{\text{eff}}(C) \in \mathbf{DM}^{\text{eff}}(k)$ (see e.g. [BVK, Proposition 2.5.5]) and shown that

$$M_{\text{tr}}^{\text{eff}}(C) \simeq \mathbb{Q} \oplus (\text{Jac}(C)_{\mathbb{Q}}^{\text{tr}}) \oplus \mathbb{Q}(1)[2].$$

The same argument works for a geometrically connected curve with a 0-cycle of degree 1; such a cycle exists because we use rational coefficients. By Proposition 2.4, we have $J(C)^{\text{tr}} \simeq a^{\text{tr}} o^{\text{tr}} J(C)^{\text{tr}} \simeq a^{\text{tr}} \varrho^{\text{tr}} J(C)^{\text{tr}} \simeq a^{\text{tr}} J(C)$ (ϱ^{tr} preserves \mathbb{A}^1 -equivalences [Ayo14b, Lemme 2.111]). Applying $\Sigma_{\text{tr}}^{\infty}$ and using that a^{tr} commutes with suspension, we get

$$M_{\text{tr}}(C) \simeq \mathbb{Q} \oplus a^{\text{tr}} \Sigma^{\infty}(\text{Jac}(C)_{\mathbb{Q}}) \oplus \mathbb{Q}(1)[2]$$

in $\mathbf{DM}(k)$. The adjunction $a^{\text{tr}} : \mathbf{DA}(k) \rightleftarrows \mathbf{DM}(k) : o^{\text{tr}}$ is an equivalence of categories by [CD, Corollary 16.2.22]. This implies that $o^{\text{tr}} M_{\text{tr}}(C) \simeq o^{\text{tr}} a^{\text{tr}} M(C) \simeq M(C)$ and similarly $o^{\text{tr}} \mathbb{Q} \simeq \mathbb{Q}$ and $o^{\text{tr}} \mathbb{Q}(1)[2] \simeq \mathbb{Q}(1)[2]$. Applying o^{tr} to the isomorphism above, we thus get an isomorphism

$$M(C) \simeq \mathbb{Q} \oplus \Sigma^{\infty}(\text{Jac}(C)_{\mathbb{Q}}) \oplus \mathbb{Q}(1)[2]$$

as required.

For a general k , the result follows from the perfect case by separation, continuity for $\mathbf{DA}(-)$ and Proposition 2.2. \square

We also need an alternative description of the motive $\Sigma^{\infty}(\mathbb{G}_m \otimes \mathbb{Q})$ (a relative, rational version of the standard description of the motivic complex $\mathbb{Z}(1)$).

Proposition 2.6. *There is a canonical isomorphism*

$$u_S : \Sigma^{\infty}(\mathbb{G}_m \otimes \mathbb{Q}) \xrightarrow{\sim} \mathbb{Q}_S(1)[1]$$

in $\mathbf{DA}(S)$. The isomorphism u_S is compatible with pullbacks and the isomorphisms R_f of Corollary 2.2: for $f : T \rightarrow S$, the diagram

$$\begin{array}{ccc} f^* \Sigma^{\infty}(\mathbb{G}_{m,S} \otimes \mathbb{Q}) & \xrightarrow[\sim]{R_f} & \Sigma^{\infty}(\mathbb{G}_{m,T} \otimes \mathbb{Q}) \\ u_S \downarrow \sim & & \downarrow u_T \\ f^*(\mathbb{Q}_S(1)[1]) & \xrightarrow{\sim} & \mathbb{Q}_T(1)[1] \end{array}$$

commutes.

Proof. By Theorem [AHPL14, Theorem 3.3] in the special case $G = \mathbb{G}_m$ (with the “Kimura dimension” $\text{kd}(\mathbb{G}_m/S)$ of the statement equal to 1), there is an isomorphism

$$\Psi := \Psi_{\mathbb{G}_m/S} : M_S(\mathbb{G}_m) \simeq \mathbb{Q} \oplus \Sigma^{\infty}(\mathbb{G}_m \otimes \mathbb{Q}).$$

It is compatible with pullbacks and the isomorphisms R_f of Corollary 2.2 (This is the precise meaning of “compatible with pullbacks” in loc.cit). By definition, $\mathbb{Q}_S(1)[1]$ is the reduced motive of $M_S(\mathbb{G}_m)$ pointed at the unit section of \mathbb{G}_m , and it follows from the naturality of $\Psi_{G/S}$ applied to the neutral section in G that the direct factor $\mathbb{Q}(1)[1]$ corresponds to the direct factor $\Sigma^{\infty}(\mathbb{G}_m \otimes \mathbb{Q})$. This yields an isomorphism $\tilde{\Psi} : \mathbb{Q}_S(1)[1] \simeq \Sigma^{\infty}(\mathbb{G}_m \otimes \mathbb{Q})$, and we put $u_S := \tilde{\Psi}^{-1}$. \square

Remark 2.7. Various results and constructions in this paper would be somewhat simplified if we knew the effective analogue of the proposition, i.e., that the natural map $\mathbb{Q}(1) \rightarrow \mathbb{G}_m[-1] \otimes \mathbb{Q}$ in $\mathbf{DA}^{\text{eff}}(S)$ is an isomorphism. The corresponding statement in $\mathbf{DM}^{\text{eff}}(S)$ is known if S is normal [CD, Proposition 11.2.11.].

We also need a version with transfers of this statement.

Corollary 2.8. *Assume S is excellent. There is a canonical isomorphisms*

$$u_S^{\text{tr}} : \Sigma_{\text{tr}}^{\infty} \mathbb{G}_m^{\text{tr}} \otimes \mathbb{Q} \xrightarrow{\sim} \mathbb{Q}_S(1)[1]$$

It is compatible with pullbacks in the same way as in Proposition 2.6. Modulo the isomorphism of Proposition 2.4, we have in fact

$$a_{\text{tr}} u_S = u_S^{\text{tr}}.$$

Proof. For our purposes, it is enough to define u_S^{tr} as $a_{\text{tr}}u_S$ modulo the isomorphism of Proposition 2.4 (which holds under the assumption that S is excellent). The claims then follow from Proposition 2.6. \square

Corollary 2.9. *Let T/S be a torus, and $X_*(T)$ its cocharacter lattice. There is an isomorphism*

$$\Sigma^\infty T_{\mathbb{Q}} \simeq \Sigma^\infty X_*(T)_{\mathbb{Q}}(1)[1].$$

In particular, if S is geometrically unibranch, the motive $\Sigma^\infty T_{\mathbb{Q}}$ is in $\mathbf{DA}_{1,c}^{\text{gsm}}(S)$.

Proof. In this proof, we distinguish between derived and underived tensor products for clarity. There is a natural morphism $X_*(T) \otimes_{\mathbb{G}_m} T$ of étale sheaves on \mathbf{Sm}/S , which is an isomorphism (this can be checked étale locally, hence for a split torus, where it is obvious). Since the functor Σ^∞ is monoidal, we have $\Sigma^\infty(X_*(T)_{\mathbb{Q}} \otimes (\mathbb{G}_m \otimes \mathbb{Q})) \simeq \Sigma^\infty(X_*(T)_{\mathbb{Q}}) \otimes \Sigma^\infty(\mathbb{G}_m \otimes \mathbb{Q}) \simeq \Sigma^\infty X_*(T)_{\mathbb{Q}}(1)[1]$ (by Proposition 2.6). It remains to check that the tensor product $X_*(T) \otimes_{\mathbb{G}_m} T$ coincides with the derived tensor product; this follows from the fact that the lattice $X_*(T)$ is étale locally free, hence flat.

If S is geometrically unibranch, $X_*(T)_{\mathbb{Q}}$ is a direct factor of the sheaf $\mathbb{Q}(V)$ for V/S finite étale by Lemma A.2, so it is geometrically smooth. \square

Remark 2.10. For more precise (integral) results on motives attached to tori over a field, see [HK06, §7].

We can now give a result which is our main source of compact homological 1-motives.

Proposition 2.11. *Let G be a smooth commutative group scheme over S . Then $\Sigma^\infty G_{\mathbb{Q}}$ lies in $\mathbf{DA}_{1,c}(S)$.*

Proof. Write $M = \Sigma^\infty G_{\mathbb{Q}}$. By [AHPL14, Theorem 3.3.(3)], M is a compact motive. It remains to show that M is an homological 1-motive. The proof of [AHPL14, Theorem 3.3.(3)] essentially shows this as well, but we provide an argument for convenience. By compactness and Proposition 1.23, it is enough to show that for all $s \in S$, $s^* \Sigma^\infty M$ is in $\mathbf{DA}_1(s)$. By Proposition 2.2 (in the case $K_* = G[0]$), continuity for $\mathbf{DA}_1(-)$ (Proposition 1.21) and separation, we are reduced to the case where S is the spectrum of a perfect field k .

The group scheme G over the field k has a neutral component G° which is smooth and of finite type. The quotient group scheme G/G° is a discrete group scheme so its motive lies in $\mathbf{DA}_0(k) \subset \mathbf{DA}_1(k)$. In the case of a smooth commutative connected algebraic group, we reduce by a standard dévissage to the cases of unipotent algebraic groups, tori and abelian varieties.

A unipotent algebraic group over a perfect field has a composition series with \mathbb{G}_a factors, and the motive $\Sigma^\infty \mathbb{G}_a \otimes \mathbb{Q}$ is trivial by [AEWH15, Lemma 7.4.5] (proved in $\mathbf{DM}^{\text{eff}}(k)$, this yields the result in $\mathbf{DA}(k)$ by applying $\Sigma^\infty \alpha_{\text{tr}}$). If $G = T$ is a torus, let $e : \mathbf{Spec}(l) \rightarrow \mathbf{Spec}(k)$ be a finite étale morphism with T_l split. Then $e^* \Sigma^\infty(T \otimes \mathbb{Q}) \simeq \Sigma^\infty(T_l \otimes \mathbb{Q})$ (this is easy because e is smooth). By a transfer argument using [Ayo07a, Lemme 2.1.165] and Proposition 1.13 (ii), this reduces us to the case of split tori, and then by direct sum to the case of \mathbb{G}_m , which follows from Proposition 2.6. If $G = A$ is an abelian variety, using [Kat99, Theorem 11] reduces the case of A to the case of a Jacobian $J(C)$ of a smooth projective curve C/k with a rational point. The fact that $\Sigma^\infty(J(C) \otimes \mathbb{Q})$ is in $\mathbf{DA}_1(k)$ follows from Proposition 2.5. \square

We now lay the technical groundwork for the study of the motivic Picard functor in Section 3.3. Let $n \in \mathbb{N}$. Recall that there is an adjunction

$$\text{Sus}^n : \mathbf{DA}^{\text{eff}}(S) \rightleftarrows \mathbf{DA}(S) : \text{Ev}_n$$

with $\text{Sus}^0 = \Sigma^\infty$ and for $m \in \mathbb{N}$ a canonical isomorphism

$$\text{Sus}^n(M) \simeq \Sigma^\infty M(-n)[-2n] \in \mathbf{DA}(S).$$

Using the map $u_S : \Sigma^\infty(\mathbb{G}_m \otimes \mathbb{Q}) \rightarrow \mathbb{Q}_S(1)[1]$, we get a map

$$\text{Sus}^1(\mathbb{G}_m \otimes \mathbb{Q}[1]) \rightarrow \mathbb{Q}_S$$

which by adjunction corresponds to a map

$$w_S : \mathbb{G}_m \otimes \mathbb{Q}[1] \rightarrow \text{Ev}_1(\mathbb{Q}_S).$$

If S is excellent, there is an analogous construction for motives with transfers (using the map u_S^{tr} instead of u_S), resulting in a map

$$w_S^{\text{tr}} : \mathbb{G}_m^{\text{tr}} \otimes \mathbb{Q}[1] \rightarrow \text{Ev}_1^{\text{tr}}(\mathbb{Q}_S^{\text{tr}})$$

in $\mathbf{DM}^{\text{eff}}(S)$.

Let $f : X \rightarrow S$ be a morphism of schemes. To state the compatibility of w_S with base change, we introduce the composition

$$d_f : f^* \text{Ev}_1^1 \mathbb{Q}_S \xrightarrow{\epsilon} \text{Ev}_1 \text{Sus}^1 f^* \text{Ev}_1 \mathbb{Q}_S \simeq \text{Ev}_1 f^* \text{Sus}^1 \text{Ev}_1 \mathbb{Q}_S \xrightarrow{\eta} \text{Ev}_1 f^* \mathbb{Q}_S \simeq \text{Ev}_1 f^* \mathbb{Q}_X$$

where the isomorphism in the middle is the canonical isomorphism $\text{Sus}^1 f^* \simeq f^* \text{Sus}^1$.

Proposition 2.12. *Let S be a noetherian finite-dimensional scheme.*

(i) *If $f : X \rightarrow S$ is any morphism of finite type, the following diagram*

$$\begin{array}{ccc} f^*(\mathbb{G}_m \otimes \mathbb{Q}[1]) & \xrightarrow[\sim]{R_f} & \mathbb{G}_m \otimes \mathbb{Q}[1] \\ f^* w_S \downarrow \sim & & \downarrow w_X \\ f^* \text{Ev}_1 \mathbb{Q}_S & \xrightarrow[d_f]{} & \text{Ev}_1 \mathbb{Q}_X \end{array}$$

commutes.

(ii) *Assume S is regular. Then the morphism w_S is an isomorphism.*

(iii) *If $f : X \rightarrow S$ is a morphism of finite type with X and S regular, then d_f is an isomorphism.*

Let S be a regular scheme. The morphism w_S is an isomorphism. , so that d_f is an isomorphism when X is also regular.

Proof. Going through the definitions of w_S and d_f , we see that the diagram in (i) is obtained from the commutative diagram of Proposition 2.6 via the adjunction $\text{Sus}^1 \dashv \text{Ev}_1$ and the commutation of Sus^1 and f^* . This shows (i). Since (iii) clearly follows from the combination of (i) and (ii), we are left with proving (ii).

Since $\mathbf{DA}^{\text{eff}}(S)$ is generated as a triangulated category by objects of the form $M_S^{\text{eff}}(X)[n]$ for $f : X \rightarrow S \in \mathbf{Sm}/S$ and $n \in \mathbb{Z}$, it is enough to show that for such an object, the induced map

$$\mathbf{DA}^{\text{eff}}(S)(M_S^{\text{eff}}(X)[n], \mathbb{G}_m \otimes \mathbb{Q}[1]) \xrightarrow{w_S^*} \mathbf{DA}^{\text{eff}}(S)(M_S^{\text{eff}}(X)[n], \text{Ev}_1(\mathbb{Q}_S))$$

is an isomorphism. The idea is to compare both sides to similar morphisms in the derived category $D(\mathbf{Sm}/S)$. Consider the following diagram.

$$\begin{array}{ccccc} D(\mathbf{Sm}/S)(\mathbb{Q}_S(X)[n], \mathbb{G}_m[1]) & \xrightarrow{(\alpha)} & \mathbf{DA}^{\text{eff}}(S)(M_S^{\text{eff}}(X)[n], \mathbb{G}_m[1]) & \xrightarrow{w_S^*} & \mathbf{DA}^{\text{eff}}(S)(M_S^{\text{eff}}(X)[n], \text{Ev}_1(\mathbb{Q}_S)) \\ \downarrow \sim \text{adj} & & \downarrow \Sigma^\infty & & \downarrow \text{adj} \sim \\ & & \mathbf{DA}(S)(M_S(X)[n], \Sigma^\infty(\mathbb{G}_m)[1]) & \xrightarrow{u_{S*}} & \mathbf{DA}(S)(M_S(X)[n], \mathbb{Q}_S(1)[2]) \\ & & \downarrow \sim \text{adj} & & \downarrow \text{adj} \sim \\ D(\mathbf{Sm}/X)(\mathbb{Q}_X[n], f^* \mathbb{G}_m[1]) & & \mathbf{DA}(X)(\mathbb{Q}_X[n], f^* \Sigma^\infty \mathbb{G}_m[1]) & \xrightarrow{(f^*(u_S))^*} & \mathbf{DA}(X)(\mathbb{Q}_X[n], \mathbb{Q}_X(1)[2]) \\ \downarrow \sim R_{f*} & & \downarrow \sim R_{f*} & & \parallel \\ D(\mathbf{Sm}/X)(\mathbb{Q}_X[n], \mathbb{G}_m[1]) & \longrightarrow & \mathbf{DA}(X)(\mathbb{Q}_X[n], \Sigma^\infty \mathbb{G}_m[1]) & \xrightarrow[u_{X*}]{\sim} & \mathbf{DA}(X)(\mathbb{Q}_X[n], \mathbb{Q}_X(1)[2]) \end{array}$$

(β)

The square (A) commutes because the isomorphisms R_f in the derived category and in \mathbf{DA} are compatible by construction. The square (B) commutes by construction of w_S and u_S . The square (C) commutes by naturality of adjunction. Finally, the square (D) commutes by Proposition 2.6.

To complete the proof that w_{S*} is an isomorphism, it remains to see that the maps (α) and (β) are isomorphisms as well. For (β), this is precisely the statement of Proposition B.6 (ii)-(iv). Let us prove that (α) is an isomorphism.

Since S is regular, all smooth S -schemes are regular. They are in particular reduced, which implies that \mathbb{G}_m is \mathbb{A}^1 -invariant on \mathbf{Sm}/S , and normal, which implies that $\mathrm{Pic} = H^1(-, \mathbb{G}_m)$ is \mathbb{A}^1 -invariant. The higher cohomology groups $H^i(-, \mathbb{G}_m)$ for $i \geq 2$ are torsion on regular schemes by [Gro68, Proposition 1.4]. All together, this implies that the sheaf $\mathbb{G}_m \otimes \mathbb{Q}$ is \mathbb{A}^1 -local in the model category underlying $\mathbf{DA}^{\mathrm{eff}}(S)$. This implies that the morphism $(\alpha) : \mathbf{D}(\mathbf{Sm}/S)(\mathbb{Q}_S(X)[n], \mathbb{G}_m \otimes \mathbb{Q}[1]) \rightarrow \mathbf{DA}^{\mathrm{eff}}(M_S^{\mathrm{eff}}(X), \mathbb{G}_m \otimes \mathbb{Q})$ is an isomorphism. This completes the proof that w_S is an isomorphism. \square

2.2. Motives of Deligne 1-motives. We relate the category $\mathcal{M}_1(S)$ of Deligne 1-motives with rational coefficients (recalled in Section A) to $\mathbf{DA}(S)$. Let $\mathbb{M} = [L \rightarrow G] \otimes \mathbb{Q}$ be in $\mathcal{M}_1(S)$. Then by seeing \mathbb{M} as a complex of étale sheaves of \mathbb{Q} -vector spaces on \mathbf{Sm}/S , we can associate to \mathbb{M} an object in $\mathbf{DA}^{\mathrm{eff}}(S)$, which we also denote by \mathbb{M} .

Corollary 2.13. *Let $\mathbb{M} \in \mathcal{M}_1(S)$. Then $\Sigma^\infty(\mathbb{M})$ lies in $\mathbf{DA}_{1,c}(S)$. If S is moreover assumed to be geometrically unibranch, then the motive $\Sigma^\infty(\mathbb{M})$ is also geometrically smooth, thus lies in $\mathbf{DA}_{1,c}^{\mathrm{gsm}}(S)$.*

Proof. Write $\mathbb{M} = [L \rightarrow G] \otimes \mathbb{Q}$. We apply Proposition 2.11 to the distinguished triangle

$$\Sigma^\infty G_{\mathbb{Q}}[-1] \rightarrow \Sigma^\infty(\mathbb{M}) \rightarrow \Sigma^\infty L_{\mathbb{Q}} \xrightarrow{+}$$

which proves the first part. Assume now S to be geometrically unibranch. We have a further distinguished triangle

$$\Sigma^\infty T_{\mathbb{Q}} \rightarrow \Sigma^\infty G_{\mathbb{Q}} \rightarrow \Sigma^\infty L_{\mathbb{Q}} \xrightarrow{+}.$$

The motives $\Sigma^\infty T_{\mathbb{Q}}$ and $\Sigma^\infty L_{\mathbb{Q}}$ are geometrically smooth by Corollary 2.9 and its proof. The motive $\Sigma^\infty A_{\mathbb{Q}}$ is a direct factor of the homological motive of A by Theorem [AHPL14, Theorem 3.3], so it is geometrically smooth. This completes the proof. \square

From Corollary and the definition of Σ^∞ , we deduce the following.

Corollary 2.14. *Let $f : T \rightarrow S$ be a morphism of schemes. There is an isomorphism of functors*

$$R_f : f^* \Sigma^\infty \xrightarrow{\sim} \Sigma^\infty f^* : \mathcal{M}_1(S) \rightarrow \mathbf{DA}(T).$$

which is compatible with further pullbacks.

As explained in Section A.3, we have also a covariant functoriality for finite étale morphisms, coming from Weil restrictions of scalars. Here is how this relates to pushforwards of motives.

Lemma 2.15. *Let $f : T \rightarrow S$ be a finite étale morphism of schemes. There is an isomorphism of functors*

$$f_* \Sigma_S^\infty \xrightarrow{\sim} \Sigma_T^\infty f_*.$$

Proof. Because of the definition of pushforwards in $\mathcal{M}_1(-)$ (Definition A.17), it is enough to show the following: for G/T smooth commutative group scheme (not necessarily of finite type), there is a natural isomorphism $f_* \Sigma^\infty G_{\mathbb{Q}} \simeq \Sigma^\infty (R_f G)_{\mathbb{Q}}$ (note that here we do not claim that $R_f G$ is representable and only consider it as a sheaf of abelian groups). We have a sequence of natural isomorphisms

$$\begin{aligned} f_* \Sigma^\infty G_{\mathbb{Q}} &\simeq f_{\sharp} \Sigma^\infty G_{\mathbb{Q}} \\ &\simeq \Sigma^\infty f_{\sharp} G_{\mathbb{Q}} \\ &\simeq \Sigma^\infty f_* G_{\mathbb{Q}} \\ &\simeq \Sigma^\infty \underline{f}_* G_{\mathbb{Q}} \\ &\simeq \Sigma^\infty (R_f G)_{\mathbb{Q}} \end{aligned}$$

where the first and third isomorphisms follow from the fact that f is finite étale, the second comes from the commutation between Σ^∞ and f_{\sharp} , the fourth follows from the fact that \underline{f}_* in $\mathbf{DA}^{\mathrm{eff}}(-)$ preserves $(\mathbb{A}^1, \text{ét})$ -equivalences for f finite (an argument can be found Part A of the proof of [Ayo14b, Lemme B.7]), and the last is the definition of the Weil restriction of scalars. This completes the proof. \square

2.3. Picard complex. Classically the Picard functor of a morphism of schemes f is defined as $R^1 f_* \mathbb{G}_m$. We introduce a variant of this construction which includes information about relative connected components.

Definition 2.16. Let $f : X \rightarrow S$ be a finite type morphism of schemes. The Picard complex $P(X/S)$ of X over S is the object $\tau_{\geq 0} f_* (\mathbb{G}_m \otimes \mathbb{Q}[1]) \in D_{[0,1]}(\mathbf{Sm}/S)$.

Remark 2.17. Recall from [SGA73, Exposé XVIII §1.4] that there is an equivalence of categories between the category of commutative group stacks over a site \mathcal{S} (with morphisms taken up to 2-isomorphisms) and the category $D_{[0,1]}(\mathbf{Sh}(\mathcal{S}, \mathbb{Z}))$. The Picard complex corresponds via this equivalence to the smooth Picard stack, i.e., the version for \mathbf{Sm}/S of the usual Picard stack (see e.g. [Bro09]). This point of view will not be used in the rest of this paper.

We will also need a version with transfers.

Definition 2.18. Let S be an excellent scheme, $f : X \rightarrow S$ a finite type morphism of schemes. The Picard complex with transfers $P^{\text{tr}}(X/S)$ of X over S is the object $\tau_{\geq 0} f_* (\mathbb{G}_m^{\text{tr}} \otimes \mathbb{Q}[1]) \in D_{[0,1]}(\text{Cor}/S)$. There is a canonical map

$$a^{\text{tr}} P(X/S) \longrightarrow P^{\text{tr}}(X/S)$$

coming from adjunction and [AHPL14, Proposition 2.10] in the case of \mathbb{G}_m (which applies since X is also excellent).

Since we are only interested in the rational coefficient situation, the following result will be useful.

Lemma 2.19. *Let $f : X \rightarrow S$ be a smooth morphism with S regular. Then for $i \geq 1$, the sheaf $R^i f_* (\mathbb{G}_m \otimes \mathbb{Q}[1]) \simeq R^{i+1}(\mathbb{G}_m \otimes \mathbb{Q})$ is trivial. As a consequence, we have*

$$P^{(\text{tr})}(X/S) \xrightarrow{\sim} f_* (\mathbb{G}_m^{(\text{tr})} \otimes \mathbb{Q}[1]).$$

Proof. This follows from the fact that for a regular scheme T and $i \geq 2$, the étale cohomology groups $H^i(T, \mathbb{G}_m)$ are torsion [Gro68, Proposition 1.4]. \square

We proceed to analyse the structure of $P(X/S)$, following closely and using the standard structure theory for the Picard scheme [Kle05] and the Picard stack [Bro09]. We will see that restricting to the smooth site discards quite a bit of information and leads to simpler structure.

In the sequel, we consider étale sheaves of abelian groups and \mathbb{Q} -vector spaces on the two sites Sch/S and \mathbf{Sm}/S . We have a morphism of sites $\zeta : \text{Sch}/S \rightarrow \mathbf{Sm}/S$. The restriction functor $\zeta_* : \mathbf{Sh}(\text{Sch}/S) \rightarrow \mathbf{Sh}(\mathbf{Sm}/S)$ is exact since both sites have the same points. We have $\zeta_* \mathbb{G}_m \simeq \mathbb{G}_m$. The functor ζ_* commutes with f_* and \underline{f}_* . By abuse of terminology, we will say that a sheaf of sets on \mathbf{Sm}/S is representable if it is isomorphic to the functor $\zeta_* X$ for X a not-necessarily smooth S -scheme; such a scheme is then not uniquely determined up to isomorphism.

Let us recall some standard results on smooth projective morphisms (the case where we will eventually study $P(X/S)$). First, by [Gro67, 17.16.3 (ii)], a smooth morphism f has sections locally in the étale topology. Second, by [Gro63, 7.8.6], a smooth projective f has a Stein factorisation

$$f : X \xrightarrow{f^\circ} \pi_0(X/S) := \mathbf{Spec}_S(\underline{f}_* \mathcal{O}_X) \xrightarrow{\pi_0(f)} S$$

with $\pi_0(f)$ finite étale and the construction of $\underline{f}_* \mathcal{O}_X$ (and hence $\pi_0(X/S)$) commutes with arbitrary base change, i.e., f is cohomologically flat in degree 0. Notice that in many treatments of the Picard scheme, the stronger hypothesis “ $\underline{f}_* \mathcal{O}_X \simeq \mathcal{O}_S$ universally” is used, but that here we want to keep track of the relative connected components.

We first look at the sheaf $\underline{f}_* (\mathbb{G}_m)$. For any $U \rightarrow S$ smooth, we have $\underline{f}_* (\mathbb{G}_m)(U) = \mathcal{O}^\times(X \times_S U) \simeq \mathcal{O}^\times(\pi_0(X \times U/U)) \simeq \mathcal{O}^\times(\pi_0(X/S) \times_S U)$. This shows that $\underline{f}_* \mathbb{G}_m$ is representable by a torus, the Weil restriction $\text{Res}_{\pi_0(f)} \mathbb{G}_m$ (see Definition A.12).

Next, we look at the classical Picard étale sheaf $\mathcal{P}ic_{X/S} := R^1 f_* \mathbb{G}_m \in \mathbf{Sh}((\text{Sch}/S)_{\text{ét}}, \mathbb{Z})$ and its smooth analogue $\mathcal{P}ic_{X/S}^{\text{sm}} \in \mathbf{Sh}((\mathbf{Sm}/S)_{\text{ét}}, \mathbb{Z})$ defined by the same formula on the smooth site. By exactness of ζ_* , we have $\zeta_* \mathcal{P}ic_{X/S} \simeq \underline{\zeta}_* \mathcal{P}ic_{X/S} \simeq \mathcal{P}ic_{X/S}^{\text{sm}}$.

Because f has sections locally in the étale topology, as a corollary of the Leray spectral sequence, we have for all $T \in \mathbf{Sm}/S$ a short exact sequence

$$(L) \quad 0 \rightarrow \text{Pic}(\pi_0(X_T/T)) \rightarrow \text{Pic}(X_T) \rightarrow \mathcal{P}ic_{X/S}^{\text{sm}}(T) \rightarrow 0.$$

The functors $\mathcal{P}ic_{X/S}$ come with natural subfunctors $\mathcal{P}ic_{X/S}^0$ and $\mathcal{P}ic_{X/S}^\tau$, the neutral component and the torsion component (i.e., elements “with a multiple in the neutral component”), which are special cases of the following general definition.

Definition 2.20. Let G be a functor $(\mathbf{Sch}/S)^{\text{op}} \rightarrow \mathbf{Ab}$. We define two group subfunctors G^0 and G^τ as follows. If $T = \mathbf{Spec}(k) \rightarrow S$ is the spectrum of an algebraically closed field k , then a point $t \in G(T)$ is in $G^0(T)$ if t is algebraically equivalent to 0 in the natural sense (i.e it can be connected to the neutral section of G by a sequence of smooth connected k -curves). The point t is in $G^\tau(T)$ iff there exists $n > 0$ with $t^n \in G^0(T)$. If X is a general object in \mathcal{S} , a point $t \in G(X)$ is in $G^0(X)$ (resp. $G^\tau(X)$) iff for all morphisms $\xi : T = \mathbf{Spec}(k) \rightarrow X$ for k algebraically closed, the restriction $\xi^*(t)$ is in $G^0(T)$ (resp. in $G^\tau(T)$).

We then define $\mathcal{P}ic_{X/S}^{\text{sm},0}$ (resp. $\mathcal{P}ic_{X/S}^{\text{sm},\tau}$) as $\zeta_* \mathcal{P}ic_{X/S}^0$ (resp. $\zeta_* \mathcal{P}ic_{X/S}^\tau$). The following is easy and well-known for $\mathcal{P}ic$ (it follows for instance from the fact that $\mathcal{P}ic$ is the étale sheafification of the “naive” Picard functor, for which the base change isomorphism is clear); the proof translates directly to $\mathcal{P}ic^{\text{sm}}$ and $P(X/S)$.

Lemma 2.21. *Let $\pi : T \rightarrow S$ a morphism of schemes. There is a natural isomorphism*

$$v_\pi : \pi^* \mathcal{P}ic_{X/S} \simeq \mathcal{P}ic_{X \times_S T/T}$$

which respects the neutral and the torsion components and which is compatible with further pullbacks and the isomorphisms R_g (i.e., a diagram like the one in Proposition 2.6 commutes). Similarly, there is natural morphism

$$v_\pi : \pi^* \mathcal{P}ic_{X/S}^{\text{sm}} \rightarrow \mathcal{P}ic_{X \times_S T/T}^{\text{sm}},$$

(resp.

$$v_\pi : \pi^* P(X/S) \simeq P(X \times_S T/T))$$

which is an isomorphism when π is smooth and is compatible with further pullbacks in the same way.

In general, the construction of $\mathcal{P}ic^{\text{sm}}$ and $P(X/S)$ does not commute with arbitrary base change, i.e., v_π is not always an isomorphism. We will see below some positive results.

We recall the following classical positive results of Grothendieck on the Picard scheme. We state the result both for the classical and the smooth context; the smooth result follows immediately by applying ζ_* .

Theorem 2.22. *Let $f : X \rightarrow S$ be a smooth projective morphism.*

- (i) [Gro95a, Theoreme 3.1] *The functor $\mathcal{P}ic_{X/S}^{(\text{sm})}$ is representable by a commutative group scheme, locally of finite type over S , that we denote $\text{Pic}_{X/S}^{(\text{sm})}$.*
- (ii) [Gro95b, Corollaire 2.3] *The functor $\mathcal{P}ic_{X/S}^{(\text{sm}),\tau}$ is representable by a projective group scheme $\text{Pic}_{X/S}^\tau$, which is an open and closed group subscheme of $\text{Pic}_{X/S}$.*

For the neutral component, the situation is more complex. We have nevertheless positive results that will be enough for us.

Theorem 2.23. *Let $f : X \rightarrow S$ be a smooth projective morphism.*

- (i) [Gro95b, Corollaire 3.2] *If S is the spectrum of a field k , then $\text{Pic}_{X/k}^0$ is representable by a projective algebraic group, with $\text{Pic}_{X/k}^{0,\text{red}} := (\text{Pic}_{X/k}^0)^{\text{red}}$ an abelian variety.*
- (ii) [Bro14, Proposition 2.15] *If $\text{Pic}_{X/S}^\tau$ is flat and the construction of $f_* \mathcal{O}_X$ commutes with base change, then $\text{Pic}_{X/S}^\tau$ is an extension of finite flat commutative group scheme by an abelian scheme. We call $\text{Pic}_{X/S}^{0,\text{red}}$ this abelian scheme.*
- (iii) *If S is the spectrum of a field k , the condition of (ii) holds and the two abelian varieties $\text{Pic}_{X/k}^{0,\text{red}}$ defined above coincide.*

Proof. The only thing to prove is (iii). Flatness and cohomological flatness automatically hold over a field. Let $G = \text{Pic}_{X/k}^\tau$ which is a commutative algebraic group with neutral component $G^0 = \text{Pic}_{X/k}^0$. By (i) we have $A := G_{\text{red}}^0$ abelian variety. By (ii) we have $0 \rightarrow B \rightarrow G \rightarrow F \rightarrow 0$ with B abelian variety and F a finite flat group scheme. Both A and B are sub-abelian varieties

of G and we must show $A = B$. Since $\text{Hom}(A, F) = 0$ because A is connected and reduced, we have $A \subset B$. Since B is connected and reduced, we have $B \subset A$. We conclude that $A = B$, as required. \square

This motivates the following definition.

Definition 2.24. We say that $f : X \rightarrow S$ is Pic-smooth if $\text{Pic}_{X/S}^\tau$ is flat and cohomologically flat in degree 0.

Remark 2.25. By Lemma 2.21 and the fact that flatness and cohomological flatness are stable by arbitrary base change, we see that Pic-smoothness is also stable by arbitrary base change.

Proposition 2.26. *Let $f : X \rightarrow S$ be a smooth projective morphism. Assume S is reduced. Then there is a dense open set $U \subset S$ such that $f \times_S U$ is Pic-smooth.*

Proof. Recall that $\text{Pic}_{X/S}^\tau$ is representable by a group scheme of finite type by Theorem 2.22 (ii). If S is reduced, generic flatness [Gro65, Corollaire 6.9.3] provides a dense open subset V of S over which $\text{Pic}_{X/S}^\tau \times_S U \simeq \text{Pic}_{X_V/V}^\tau$ is flat.

By standard results on cohomology and base change [Gro63, §7], because V is reduced, the flat morphism $\pi : P = \text{Pic}_{X_V/V}^\tau \rightarrow V$ is cohomologically flat in dimension 0 if the function $d^1 : s \in V \rightarrow H^1(P_s, \mathcal{O}_s)$ is locally constant. The function d^1 is upper semi-continuous [Gro63, Theorem 7.7.5, I], hence, locally constant on a dense open set U of S . This implies that $f \times_S U$ is Pic-smooth. \square

Proposition 2.27. *Let $f : X \rightarrow S$ be a smooth projective Pic-smooth morphism. Then $\text{Pic}_{X/S}^{\text{sm},0}$ is representable by the abelian scheme $\text{Pic}_{X/S}^{0,\text{red}}$.*

Proof. By Theorem 2.23 (ii), we have a short exact sequence of group schemes

$$0 \rightarrow \text{Pic}_{X/S}^{0,\text{red}} \rightarrow \text{Pic}_{X/S}^\tau \rightarrow F \rightarrow 0$$

with F finite flat group scheme. Let $T \in \mathbf{Sm}/S$ and $L \in \mathcal{P}ic_{X/S}^{\text{sm},0}(T) \subset \text{Pic}_{X/S}^\tau(T)$. Let $\xi : \mathbf{Spec}(k) \rightarrow T$ be a geometric point. By hypothesis, there exists a sequence of smooth curves connecting $\xi^*(L)$ to the zero section of $\text{Pic}_{X_k/k}^\tau$. Any morphism from a smooth curve over k to F_k is constant. This shows that the image of $\xi^*(L)$ in $F(k)$ is zero. Since this holds for all k and T/S is smooth, this implies that the induced morphism $T \rightarrow F$ is zero. This shows that we have an induced monomorphism $\mathcal{P}ic_{X/S}^{\text{sm},0} \rightarrow \text{Pic}_{X/S}^{0,\text{red}}$.

In the other direction, the morphism $\text{Pic}_{X/S}^{0,\text{red}} \rightarrow \text{Pic}_{X/S}^{\tau,\text{sm}}$ factors through $\mathcal{P}ic_{X/S}^{\text{sm},0}$ because abelian varieties are geometrically connected. This shows the above monomorphism is surjective and concludes the proof. \square

We now turn to the study of the Néron-Severi groups in families.

Definition 2.28. We define the Neron-Severi sheaf as the étale quotient sheaf

$$\mathcal{NS}_{X/S}^{(\text{sm})} := \mathcal{P}ic_{X/S}^{(\text{sm})} / \mathcal{P}ic_{X/S}^{(\text{sm}),\tau}.$$

The following lemma follows directly from the exactness of ζ_* and from Lemma 2.21.

Lemma 2.29. *We have a canonical isomorphism $\zeta_* \mathcal{NS}_{X/S} \simeq \mathcal{NS}_{X/S}^{\text{sm}}$, and the construction of $\mathcal{NS}_{X/S}$ (resp. $\mathcal{NS}_{X/S}^{\text{sm}}$) commutes with base change by an arbitrary morphism (resp. by a smooth morphism).*

We have also a useful simplification on a regular base.

Lemma 2.30. *Let $f : X \rightarrow S$ be Pic-smooth with S regular. Then for all $T \in \mathbf{Sm}/S$, the natural map*

$$\text{Pic}_{X/S}^{\text{sm}}(T) \otimes \mathbb{Q} / \mathcal{P}ic_{X/S}^{\text{sm},\tau}(T) \otimes \mathbb{Q} \longrightarrow \mathcal{NS}_{X/S}^{\text{sm}} \otimes \mathbb{Q}(T).$$

is an isomorphism.

Proof. It suffices to prove that in this situation, the cohomology group $H_{\text{ét}}^1(T, \text{Pic}_{X/S}^{\text{sm}, \tau} \otimes \mathbb{Q})$ vanishes. By Theorem 2.23 (ii), we have a short exact sequence

$$0 \rightarrow \text{Pic}_{X/S}^{0, \text{red}} \rightarrow \text{Pic}_{X/S}^{\tau} \rightarrow F \rightarrow 0$$

where $\text{Pic}_{X/S}^{0, \text{red}}$ is an abelian scheme and F is a finite flat commutative group scheme. Almost by definition, classes in $H^1(T, F)$ can be trivialized by passing to a finite flat cover, so by a transfer argument they are torsion and hence vanish after tensoring by \mathbb{Q} . On the other hand, since T is noetherian and regular, [Ray70, Proposition XIII 2.6.(ii)] and [Ray70, Proposition XIII 2.3.(ii)] imply that torsors under $\text{Pic}_{X/S}^{0, \text{red}}$ are torsion, which implies that $H^1(T, \text{Pic}_{X/S}^{0, \text{red}} \otimes \mathbb{Q}) = 0$. This concludes the proof. \square

We have a morphism of sites $\alpha : \mathbf{Sm}/S \rightarrow \mathbf{Et}/S$ where \mathbf{Et}/S is the small étale site of S . Put $\gamma = \alpha \circ \zeta : \mathbf{Sch}/S \rightarrow \mathbf{Et}/S$. We say that a sheaf F on \mathbf{Sm}/S (resp. \mathbf{Sch}/S) is constructible if it is in the essential image of the fully faithful functor α^* (resp. γ^*), or equivalently if the counit morphism $\alpha^* \alpha_* F \rightarrow F$ (resp. $\gamma^* \gamma_* F \rightarrow F$) is an isomorphism and $\alpha_* F$ (resp. $\gamma_* F$) is constructible (as a sheaf of \mathbb{Z} -modules, i.e., we do not require fibers to be finite abelian groups, but only to be finitely generated).

It is well known that the sheaf $\mathcal{NS}_{X/S}$ is far from being constructible; in particular, the rank of the geometric fibers (which are finitely generated abelian groups by [SGA71, Exp XIII, Thm 5.1]) is not a constructible function [BLR90, 8.4 Remark 8]. For the smooth Neron-Severi sheaf, the situation is somewhat better.

Proposition 2.31. *Let $f : X \rightarrow S$ be Pic-smooth with S regular. The sheaf $\mathcal{NS}_{X/S}^{\text{sm}} \otimes \mathbb{Q}$ is locally constant.*

Proof. We can assume S is connected, with generic point η . Fix a geometric point $\bar{\eta}$ over η . Since being locally constant is an étale local property and f is smooth, we can apply étale descent and Lemma 2.29 to reduce to the case where f has a section s .

By [SGA71, Exp XIII, Thm 5.1], the abelian group $\text{NS}(X_{\bar{\eta}})$ is finitely generated. It comes with a continuous action of the Galois group $\text{Gal}(\bar{\eta}/\eta)$ (which thus factors through a finite quotient). The ℓ -adic first Chern class yield a Galois-equivariant morphism $c_1 : \text{NS}(X_{\bar{\eta}}) \rightarrow H^2(X_{\bar{\eta}}, \mathbb{Q}_\ell(1))$ which is injective after tensoring by \mathbb{Q}_ℓ , hence also injective after tensoring by \mathbb{Q} . Moreover, since f is smooth and projective, the proper and smooth base change theorems for ℓ -adic cohomology imply that, for any codimension 1 point $s \in S$, the Galois representation on $H^2(X_{\bar{\eta}}, \mathbb{Q}_\ell(1))$ is unramified at s . By the commutative diagram [SGA71, Exp X, 7.13.10], this implies that $\text{NS}(X_{\bar{\eta}})_{\mathbb{Q}}$ is also unramified at s . Since S is regular, this equips $\text{NS}(X_{\bar{\eta}})_{\mathbb{Q}}$ with an action of the étale fundamental group of S at $\bar{\eta}$, which is none other than the unramified quotient of $\text{Gal}(\bar{\eta}/\eta)$ [SGA03, Proposition 8.2]. This implies that $\text{NS}(X_{\bar{\eta}})_{\mathbb{Q}}$ can be identified with the geometric generic fiber of a locally constant constructible étale sheaf of \mathbb{Q} -vector spaces $\mathcal{N}_{X/S}$, the *Neron-Severi lattice* of X over S .

We now define a morphism $e_S : \alpha_* \mathcal{NS}_{X/S}^{\text{sm}} \rightarrow \mathcal{N}_{X/S}$ as follows. We first define a morphism $\tilde{c}_S : \alpha_* \text{Pic}_{X/S}^{\text{sm}} \rightarrow \mathcal{N}_{X/S}$. Recall that $\alpha_* \text{Pic}_{X/S}^{\text{sm}}$ is the étale sheaf associated to the presheaf $\text{Pic}_{X/S}^{\text{sm}, \text{psh}} : V \in \mathbf{Et}/S \mapsto \text{Pic}(X \times_S V)$. Since $\mathcal{N}_{X/S}$ is an étale sheaf, defining \tilde{c}_S is equivalent to writing down a morphism $\text{Pic}_{X/S}^{\text{sm}, \text{psh}} \rightarrow \mathcal{N}_{X/S}$. Let $V \in \mathbf{Et}/S$, which we can assume connected, and \mathcal{L} be a line bundle on $X \times_S V$. Choose a factorisation $\bar{\eta} \rightarrow V_{\bar{\eta}} \rightarrow \eta$, which induces a morphism $\pi_1(V, \bar{\eta}) \rightarrow \pi_1(S, \bar{\eta})$. Using this factorisation, lift $\mathcal{L}_{\bar{\eta}}$ to a class in $\text{NS}(X_{\bar{\eta}}) \simeq \text{NS}(X_{V_{\bar{\eta}}} \times_{V_{\bar{\eta}}} \bar{\eta})$ which by construction is fixed by $\pi_1(V, \bar{\eta})$, so gives a section in $\mathcal{N}_{X/S}(V)$. This is the required class $\tilde{c}_S([\mathcal{L}])$. The morphism \tilde{c}_S is trivial on $\alpha_* \text{Pic}_{X/S}^{\text{sm}, \tau}$ since algebraic equivalence over V implies algebraic equivalence over $\bar{\eta}$. So \tilde{c}_S induces a morphism $e_S : \alpha_* \mathcal{NS}_{X/S}^{\text{sm}} \rightarrow \mathcal{N}_{X/S}$ as required.

The goal of the rest of the proof is to establish that

- a) the counit morphism $\alpha^* \alpha_* \mathcal{NS}_{X/S}^{\text{sm}} \rightarrow \mathcal{NS}_{X/S}^{\text{sm}}$ is an isomorphism, and
- b) $e_S : \alpha_* \mathcal{NS}_{X/S}^{\text{sm}} \rightarrow \mathcal{N}_{X/S}$ is an isomorphism,

which together imply the proposition. We first proceed to reduce the proof of points a) and b) to the case where S is a field, by restriction to the generic point η .

Lemma 2.32. *With the hypotheses of the proposition, assume S is moreover irreducible and denote by η the generic point of S . Then the adjunction morphism*

$$\mathcal{N}\mathcal{S}_{X/S}^{\text{sm}} \otimes \mathbb{Q} \rightarrow \eta_* \eta^* \mathcal{N}\mathcal{S}_{X/S}^{\text{sm}} \otimes \mathbb{Q}$$

is an isomorphism.

Proof. Since η is pro-smooth, we deduce from Lemma 2.29 that $\eta^* \mathcal{N}\mathcal{S}_{X/S}^{\text{sm}} \simeq \mathcal{N}\mathcal{S}_{X_\eta/\eta}^{\text{sm}}$.

Let $T \in \mathbf{Sm}/S$. One sees that the map $\mathcal{N}\mathcal{S}_{X/S}^{\text{sm}} \otimes \mathbb{Q}(T) \rightarrow \eta_* \eta^* \mathcal{N}\mathcal{S}_{X/S}^{\text{sm}} \otimes \mathbb{Q}(T)$ can be identified, modulo the previous paragraph and the identification of Lemma 2.30 with the natural map

$$\text{Pic}_{X/S}(T)_{\mathbb{Q}} / \text{Pic}_{X/S}^{\tau}(T)_{\mathbb{Q}} \longrightarrow \text{Pic}_{X_\eta/\eta}(T_\eta)_{\mathbb{Q}} / \text{Pic}_{X_\eta/\eta}^{\tau}(T_\eta)_{\mathbb{Q}}$$

which because f has a section takes the more concrete form

$$\text{Pic}(X_T)_{\mathbb{Q}} / (f_T^* \text{Pic}(T)_{\mathbb{Q}} \text{Pic}_{X/S}^{\tau}(T)_{\mathbb{Q}}) \longrightarrow \text{Pic}(X_{T_\eta})_{\mathbb{Q}} / (f_{T_\eta}^* \text{Pic}(T_\eta)_{\mathbb{Q}} \text{Pic}_{X/S}^{\tau}(T_\eta)_{\mathbb{Q}})$$

We need to show that this map is bijective. Since T is regular, isomorphism classes of lines bundles can be represented by Weil divisors. Moreover, the map $Z^1(X_T) \rightarrow Z^1(X_{T_\eta})$ has a retraction given by taking the Zariski closure of a divisor on X_{T_η} in X_T . This implies the surjectivity.

We prove the injectivity. Let $\mathcal{L} \in \text{Pic}(X_T)$ such that $\mathcal{L}_\eta \in \text{Pic}(X_{T_\eta})$ lies in $f_{T_\eta}^* \text{Pic}(T_\eta) \text{Pic}_{X/S}^{\tau}(T_\eta)$. One can find a dense open set $U \subset S$ such that \mathcal{L}_U lies in $f_{T_U}^* \text{Pic}(T_U) \text{Pic}_{X/S}^{\tau}(T_U)$, say $\mathcal{L}_U = f_{T_U}^* \mathcal{L}' \cdot x^\tau$.

In this paragraph, we reduce to the case $\mathcal{L}_U = 0$. Because $\text{Pic}_{X/S}^{\tau}$ is an abelian scheme “up to a finite flat group scheme” (by the Pic-smooth hypothesis, Theorem 2.23 (ii)) and we work tensor \mathbb{Q} , we can use the flasqueness of abelian schemes over regular schemes (see e.g. [Bha12, Proposition 4.2]) to extend x^τ to an element of $\text{Pic}_{X/S}^{\tau}(T)$. We can thus assume that x^τ is trivial and that $\mathcal{L}_U = f_{T_U}^* \mathcal{L}'$. By using the argument for surjectivity on T for \mathcal{L}' , we see that we can also assume that \mathcal{L}' is trivial.

We have now \mathcal{L}_U trivial. Let D be a Weil divisor on X_T such that $\mathcal{L} \simeq \mathcal{O}_{X_T}(D)$. By the above, $D \cap T_U = \emptyset$. Because D is a divisor and f_T has connected fibers, this implies that D contains every fiber of f_T it meets, which in turns shows that D comes from a Weil divisor on T . So \mathcal{L} is in $f_T^* \text{Pic}(T)$ and the injectivity is proven. \square

Consider the following diagram of morphisms of sites

$$\begin{array}{ccc} \mathbf{Sm}/\eta & \xrightarrow{\beta} & \mathbf{Et}/\eta \\ \eta \downarrow & & \downarrow \gamma \\ \mathbf{Sm}/S & \xrightarrow{\alpha} & \mathbf{Et}/S. \end{array}$$

Since a smooth S -scheme which is étale over η is étale over a dense open set, the base change morphism $\alpha^* \gamma_* \rightarrow \eta_* \beta^*$ is an isomorphism of functors. We have a commutative diagram

$$\begin{array}{ccccccc} \alpha^* \alpha_* \mathcal{N}\mathcal{S}_{X/S}^{\text{sm}} & \xrightarrow{\sim} & \alpha^* \alpha_* \eta_* \eta^* \mathcal{N}\mathcal{S}_{X/S}^{\text{sm}} & \xrightarrow{\sim} & \alpha^* \gamma_* \beta_* \eta^* \mathcal{N}\mathcal{S}_{X/S}^{\text{sm}} & \xrightarrow{\sim} & \eta_* \beta^* \beta_* \eta^* \mathcal{N}\mathcal{S}_{X/S}^{\text{sm}} \\ \downarrow & & \downarrow & & & & \downarrow \\ \mathcal{N}\mathcal{S}_{X/S}^{\text{sm}} & \xrightarrow{\sim} & \eta_* \eta^* \mathcal{N}\mathcal{S}_{X/S}^{\text{sm}} & \xlongequal{\quad\quad\quad} & \eta_* \eta^* \mathcal{N}\mathcal{S}_{X/S}^{\text{sm}} & & \end{array}$$

where all the maps come from pullback-pushforward adjunctions and the commutativity is formal. The fact that the two horizontal maps in the first square are isomorphisms follows from Lemma 2.32. We have $\eta^* \mathcal{N}\mathcal{S}_{X/S}^{\text{sm}} \simeq \mathcal{N}\mathcal{S}_{X_\eta/\eta}^{\text{sm}}$ by Lemma 2.29; the commutative diagram above then shows that, to prove point a), one can assume $S = \eta$. A similar argument, using the fact that the construction of e_S commute with restriction to η and that $\mathcal{N}_{X/S}$ is a lattice (so that it satisfies $\mathcal{N}_{X/S} \simeq \gamma_* \gamma^* \mathcal{N}_{X/S} \simeq \gamma_* \mathcal{N}_{X_\eta/\eta}$), shows that to prove point b) one can assume $S = \eta$ as well.

This reduces the proof of a) and b) to the case where S is the spectrum of a field k . A Galois descent argument then shows that we can assume k is separably closed, so that the sheaf $\mathcal{N}_{X/S}$ becomes constant.

A convenient feature of the field case is that the morphism of sites $\alpha : \mathbf{Sm}/S \rightarrow \mathbf{Et}/S$ admits a section $\pi_0 : \mathbf{Et}/S \rightarrow \mathbf{Sm}/S$, the functor which associates to a smooth k -scheme U the étale k -scheme $\pi_0(U/k) := \mathbf{Spec}(\mathcal{O}_U(U))$; moreover, we have a canonical isomorphism $\alpha^* \simeq \pi_{0*}$.

We prove *a*) and *b*) by examining the induced maps on points of the smooth étale site $(\mathbf{Sm}/k)_{\text{ét}}$. Let U be a smooth k -scheme and $\bar{x} \rightarrow U$ be a geometric point. Let $V = U_{\bar{x}}^{\text{hs}}$ be the strict henselisation of U at \bar{x} , considered as a smooth pro- k -scheme $\mathbf{Spec}(R)$. Then the collection of all such V gives enough point of the site \mathbf{Sm}/k . Moreover, applying π_0 , we get an étale k -scheme $\pi_0(V)$ with a map $V \rightarrow \pi_0(V)$; here $\pi_0(V)$ is the spectrum of the separable closure \tilde{k} of k in the k -algebra R . Since R is noetherian, regular and local, it is factorial, so $\text{Pic}(V) = 0$. Using the relationship of $\text{Pic}(X \times_k V)$ with Weil divisors as in the proof of Lemma 2.32, we can show that the map

$$\mathcal{NS}_{X/k}^{\text{sm}}(V) \simeq \text{Pic}(X \times_k V) / \text{Pic}_{X/k}^{\tau}(V) \rightarrow \text{Pic}(X \times_k \pi_0(V)) / \text{Pic}_{X/k}^{\tau}(\pi_0(V)) \simeq \mathcal{NS}_{X/k}^{\text{sm}}(\pi_0(V))$$

is an isomorphism, proving *a*). Similarly $\text{Pic}(\pi_0(V)) = 0$. From this, one deduces that the group $\mathcal{NS}_{X/k}^{\text{sm}}(\pi_0(V))$ is isomorphic to the Neron-Severi group $\text{NS}(X_{\bar{x}})$. Combined with *a*), this proves *b*) and completes the proof. \square

This result has several useful corollaries.

Corollary 2.33. *Assume S is regular. Let $f : X \rightarrow S$ be a smooth projective Pic-smooth morphism of schemes. Then we have distinguished triangles*

$$\Sigma^{\infty}(\underline{f}_* \mathbb{G}_m \otimes \mathbb{Q})[1] \rightarrow \Sigma^{\infty} P(X/S) \rightarrow \Sigma^{\infty}(\mathcal{Pic}_{X/S}^{\text{sm}} \otimes \mathbb{Q}) \xrightarrow{+}$$

and

$$\Sigma^{\infty}(\mathcal{Pic}_{X/S}^{\text{sm}, \tau} \otimes \mathbb{Q}) \rightarrow \Sigma^{\infty}(\mathcal{Pic}_{X/S}^{\text{sm}} \otimes \mathbb{Q}) \rightarrow \Sigma^{\infty} \mathcal{NS}_{X/S}^{\text{sm}} \otimes \mathbb{Q} \xrightarrow{+}.$$

and the motive $\Sigma^{\infty} P(X/S)$ is in $\mathbf{DA}_{1,c}^{\text{gsm}}(S)$.

Proof. The distinguished triangles follow from the corresponding triangles in the derived category of sheaves. The sheaf $\underline{f}_* \mathbb{G}_m \simeq \text{Res}_{\pi_0(f)} \mathbb{G}_m$ is representable by a torus, the sheaf $\mathcal{Pic}_{X/S}^{\text{sm}, \tau}$ is representable by the abelian scheme $\text{Pic}_{X/S}^{0, \text{red}}$ because f is Pic-smooth (Theorem 2.23 (ii)), and the sheaf $\mathcal{NS}_{X/S}^{\text{sm}}$ is representable by a lattice by Proposition 2.31. We conclude using Corollary 2.13. \square

Another important corollary is the comparison with the theory with transfers.

Corollary 2.34. *Let S be a regular excellent scheme and $f : X \rightarrow S$ a smooth projective Pic-smooth morphism. We have distinguished triangles*

$$(\underline{f}_* \mathbb{G}_m^{\text{tr}} \otimes \mathbb{Q}) \rightarrow P(X/S)_{\mathbb{Q}}^{\text{tr}} \rightarrow (\mathcal{Pic}_{X/S}^{\text{sm}, \text{tr}} \otimes \mathbb{Q}) \xrightarrow{+}$$

and

$$\mathcal{Pic}_{X/S}^{\text{sm}, \tau, \text{tr}} \rightarrow (\mathcal{Pic}_{X/S}^{\text{sm}, \text{tr}} \otimes \mathbb{Q}) \rightarrow \mathcal{NS}_{X/S}^{\text{sm}, \text{tr}} \otimes \mathbb{Q} \xrightarrow{+}.$$

Moreover, the natural map

$$a^{\text{tr}} P(X/S) \longrightarrow P^{\text{tr}}(X/S)$$

is an isomorphism.

Proof. The distinguished triangles follow from the same arguments as for $P(X/S)$, and each term of the triangles is represented by a smooth commutative group scheme. The result then follows from [AHPL14, Proposition 2.10] \square

Finally, we look more closely at the case of a relative smooth projective curve, where things are simpler.

Proposition 2.35. *Let $f : C \rightarrow S$ be a smooth projective relative curve (S arbitrary). Then f is Pic-smooth, and $\mathcal{NS}_{C/S}$ is represented by a lattice canonically isomorphic to $\mathbb{Q}[\pi_0(C/S)]$. In particular, for any $g : T \rightarrow S$, the morphism $v_g : g^* P(C/S) \rightarrow P(C_T/T)$ is an isomorphism.*

Proof. When f has connected fibers, this is contained in the computation of relative Picard schemes for smooth projective curves in [BLR90, Theorem 9.3.1]. Since $\pi_0(C/S)$ is finite étale, the general case follows by étale descent. The addendum comes from the fact that the construction of $\pi_0(C/S)$ commutes with arbitrary base change. \square

We also adopt a more traditional notation in this special case.

Notation 2.36. Let $f : C \rightarrow S$ be a smooth projective relative curve. We call the abelian scheme $\text{Pic}^{0,\text{red}}(C/S)$ the (relative) *Jacobian* of C over S , and we denote it by $\text{Jac}(C/S)$.

Let $f : X \rightarrow S$ be a finite type morphism of schemes. We introduce a morphism $\Theta_f : \Sigma^\infty \mathbf{P}(X/S)(-1)[-2] \rightarrow f_* \mathbb{Q}_X$ which plays a key role in the next section.

We start with the adjunction morphism

$$\text{Sus}^1 \text{Ev}_1 f_* \mathbb{Q}_X \xrightarrow{\eta} f_* \mathbb{Q}_X.$$

The functors Ev_1 and f_* commute, because they are right derived functors of right Quillen functors which commute at the model category level. We thus have a canonical isomorphism

$$f_* \text{Ev}_1 \simeq \text{Ev}_1 f_* : \mathbf{DA}^{\text{eff}}(X) \longrightarrow \mathbf{DA}(S).$$

By composition we obtain a map

$$\text{Sus}^1 f_* \text{Ev}_1(\mathbb{Q}_X) \longrightarrow f_* \mathbb{Q}_X.$$

We then use the morphism w_S to obtain a map

$$\text{Sus}^1 f_*(\mathbb{G}_m \otimes \mathbb{Q}[1]) \longrightarrow f_* \mathbb{Q}_X.$$

Recall that $\text{Sus}^1 \simeq \Sigma^\infty(-)(-1)[-2]$ so that we get a morphism

$$\Sigma^\infty f_*(\mathbb{G}_m \otimes \mathbb{Q}[1])(-1)[-2] \longrightarrow f_* \mathbb{Q}_X.$$

Composing with the adjunction morphism $\tau_{\geq 0}(-) \rightarrow \text{id}$ provides the desired morphism

$$\Theta_f : \Sigma^\infty \mathbf{P}(X/S)(-1)[-2] \longrightarrow f_* \mathbb{Q}_X.$$

We can do the same construction in $\mathbf{DM}(-)$ using w_S^{tr} , resulting in a morphism

$$\Theta_f^{\text{tr}} : \Sigma_{\text{tr}}^\infty \mathbf{P}^{\text{tr}}(X/S)(-1)[-2] \longrightarrow f_* \mathbb{Q}_X^{\text{tr}}$$

in $\mathbf{DM}(S)$. Later on, we need some alternative descriptions of the map Θ_f^{tr} .

Proposition 2.37. *Let S be a regular scheme and f be a smooth projective Pic-smooth morphism. Then the natural morphism $a^{\text{tr}} \Sigma^\infty \mathbf{P}(X/S) \simeq \Sigma_{\text{tr}}^\infty a^{\text{tr}} \mathbf{P}(X/S) \rightarrow \Sigma^\infty \mathbf{P}^{\text{tr}}(X/S)$ is an isomorphism by Corollary 2.34, and the natural morphism $a^{\text{tr}} f_* \mathbb{Q}_X \rightarrow f_* \mathbb{Q}_X^{\text{tr}}$ in $\mathbf{DM}(S)$ is an isomorphism because of the comparison theorem between \mathbf{DA} and \mathbf{DM} on geometrically unibranch schemes [CD, 16.2.22]. Modulo these identifications, we have*

$$a^{\text{tr}} \Theta_f = \Theta_f^{\text{tr}}.$$

Moreover, the morphism Θ_f^{tr} admits the following alternative description. The morphism $\alpha_G^{\text{eff},\text{tr}} : \mathbb{Q}(1)[1] \rightarrow \mathbb{G}_m \otimes \mathbb{Q}$ is an isomorphism in $\mathbf{DM}^{\text{eff}}(X)$ by [CD, Proposition 11.2.1], and we denote by $u_X^{\text{eff},\text{tr}}$ its inverse, so that we have $\Sigma^\infty u_X^{\text{eff},\text{tr}} = u_X^{\text{tr}}$ (inverses to the same map). Then Θ_f^{tr} is the composition

$$\text{Sus}^1 f_* \mathbb{G}_m \otimes \mathbb{Q}[1] \xrightarrow{\text{Sus}^1 f_* u_X^{\text{eff},\text{tr}}} \text{Sus}^1 f_*(\mathbb{Q}^{\text{tr}}(1)[2]) \simeq (\Sigma^\infty f_*(\mathbb{Q}^{\text{tr}}(1)[2])(-1)[-2] \rightarrow f_* \mathbb{Q}^{\text{tr}})$$

where the last morphism is the natural transformation $\Sigma^\infty f_* \rightarrow f_* \Sigma^\infty$.

Proof. The first statement translates into proving the commutativity of the outer square in the following diagram.

$$\begin{array}{ccccccc} a^{\text{tr}} \text{Sus}^1 f_* \mathbb{G}_m \otimes \mathbb{Q}[1] & \xrightarrow{w_X} & a^{\text{tr}} \text{Sus}^1 f_* \text{Ev}_1 \mathbb{Q} & \xrightarrow{\sim} & a^{\text{tr}} \text{Sus}^1 \text{Ev}_1 f_* \mathbb{Q} & \xrightarrow{\eta} & a^{\text{tr}} f_* \mathbb{Q} \\ \sim \downarrow & & \sim \downarrow & & \sim \downarrow & & \downarrow \\ \text{Sus}_{\text{tr}}^1 a^{\text{tr}} f_* \mathbb{G}_m \otimes \mathbb{Q}[1] & \xrightarrow{w_X} & \text{Sus}_{\text{tr}}^1 a^{\text{tr}} f_* \text{Ev}_1 \mathbb{Q} & \longrightarrow & \text{Sus}_{\text{tr}}^1 a^{\text{tr}} \text{Ev}_1 f_* \mathbb{Q} & & \\ \sim \downarrow & & \downarrow & & \downarrow & & \\ \text{Sus}_{\text{tr}}^1 f_* a^{\text{tr}} \mathbb{G}_m \otimes \mathbb{Q}[1] & \xrightarrow{w_X} & \text{Sus}_{\text{tr}}^1 f_* a^{\text{tr}} \text{Ev}_1 \mathbb{Q} & \longrightarrow & \text{Sus}_{\text{tr}}^1 \text{Ev}_1^{\text{tr}} a^{\text{tr}} f_* \mathbb{Q} & & \\ \sim \downarrow & & \downarrow & & \downarrow & & \\ \text{Sus}_{\text{tr}}^1 f_* \mathbb{G}_m^{\text{tr}} \otimes \mathbb{Q}[1] & \xrightarrow{w_X} & \text{Sus}_{\text{tr}}^1 f_* \text{Ev}_1^{\text{tr}} \mathbb{Q}^{\text{tr}} & \xrightarrow{\sim} & \text{Sus}_{\text{tr}}^1 \text{Ev}_1^{\text{tr}} f_* \mathbb{Q}^{\text{tr}} & \xrightarrow{\eta} & f_* \mathbb{Q}^{\text{tr}} \end{array}$$

All squares in this diagram commute either by naturality of adjunctions or because of the commutation $\mathrm{Sus}^1_{\mathrm{tr}} a^{\mathrm{tr}} \simeq a^{\mathrm{tr}} \mathrm{Sus}^1$ and some easy adjunction arguments.

For the second statement, we observe that Θ_f^{tr} is defined as the composition

$$\mathrm{Sus}^1 f_* \mathbb{G}_{\mathrm{m}}^{\mathrm{tr}}[1] \xrightarrow{\epsilon} \mathrm{Sus}^1 f_* \mathrm{Ev}_* \mathrm{Sus}^1 \mathbb{G}_{\mathrm{m}}^{\mathrm{tr}}[1] \xrightarrow{\mathrm{Sus}^1 f_* \mathrm{Ev}_1 (u_X^{\mathrm{tr}}(-1)[-2])} \mathrm{Sus}^1 f_* \mathrm{Ev}_1 \mathbb{Q}^{\mathrm{tr}} \simeq \mathrm{Sus}^1 \mathrm{Ev}_1 f_* \mathbb{Q}^{\mathrm{tr}} \xrightarrow{\eta} f_* \mathbb{Q}^{\mathrm{tr}}$$

(we have expanded the definition of w_X^{tr}), whereas the map of the statement is the composition

$$\mathrm{Sus}^1 f_* \mathbb{G}_{\mathrm{m}}^{\mathrm{tr}}[1] \xrightarrow{\mathrm{Sus}^1 f_* u_X^{\mathrm{eff}, \mathrm{tr}}} \mathrm{Sus}^1 f_* \mathbb{Q}^{\mathrm{tr}}(1)[2] \xrightarrow{\epsilon} \mathrm{Sus}^1 f_* \mathrm{Ev}_1 \mathrm{Sus}^1 \mathbb{Q}^{\mathrm{tr}}(1)[2] \simeq \mathrm{Sus}^1 \mathrm{Ev}_1 f_* \mathrm{Sus}^1 \mathbb{Q}^{\mathrm{tr}}(1)[2] \xrightarrow{\eta} f_* \mathbb{Q}^{\mathrm{tr}}$$

(we have expanded the definition of $\Sigma^\infty f_* \rightarrow f_* \Sigma^\infty$). The equality of those two compositions follows from naturality of the $(\mathrm{Sus}^1, \mathrm{Ev}_1)$ adjunction and the equality $\mathrm{Sus}^1 u_X^{\mathrm{eff}, \mathrm{tr}} = u_X^{\mathrm{tr}}(-1)[-2]$. \square

We finish with a study of the compatibility of the map Θ_f with base change.

Proposition 2.38. *Let $f : X \rightarrow S$ be a smooth projective Pic-smooth morphism of schemes. Let $g : T \rightarrow S$ be any morphism. Let $f' : X_T \rightarrow T$ be the pullback (which is still smooth projective Pic-smooth by Remark 3.3). The following diagram commutes in $\mathbf{DA}(S)$.*

$$\begin{array}{ccc} g^* \Sigma^\infty P(X/S)(-1)[-2] & \xrightarrow{g^* \Theta_f} & g^* f_* \mathbb{Q}_X \\ \nu_g \circ R_g \downarrow & & \downarrow \mathrm{Ex}_*^* \\ \Sigma^\infty P(X_T/T)(-1)[-2] & \xrightarrow{\Theta_{f'}} & f'_* \mathbb{Q}_{X_T} \end{array}$$

In particular, if ν_g and Ex_*^* are isomorphisms, then

$$g^* \Theta_f \text{ isomorphism} \Leftrightarrow \Theta_{f'} \text{ isomorphism}$$

Proof. The first observation is that, using the naturality of the natural transformation $g^* \tau_{\geq 0} \rightarrow \tau_{\geq 0} g^*$, we can reduce to the same commutation for the full $f_* \mathbb{G}_{\mathrm{m}} \otimes \mathbb{Q}[1]$ instead of $P(X/S)$.

In the rest of the proof, we need notations for the natural transformations

$$(\alpha_f) : f^* \mathrm{Sus}^1 \xrightarrow{\sim} \mathrm{Sus}^1 f^*$$

$$(\beta_f) : f_* \mathrm{Ev}_1 \xrightarrow{\sim} \mathrm{Ev}_1 f_*$$

and

$$(\gamma_f) : f^* \mathrm{Ev}_1 \longrightarrow \mathrm{Ev}_1 f^*.$$

The natural isomorphisms (α) , (β) are derived versions of isomorphisms at the level of model categories of spectra. The natural transformation (γ) can be defined in two equivalent ways, one using (α) and one using (β) : namely, as the two equal compositions

$$f^* \mathrm{Ev}_1 \xrightarrow{\epsilon} \mathrm{Ev}_1 \mathrm{Sus}^1 f^* \mathrm{Ev}_1 \xrightarrow{(\alpha_f^{-1})} \mathrm{Ev}_1 f^* \mathrm{Sus}^1 \mathrm{Ev}_1 \xrightarrow{\eta} \mathrm{Ev}_1 f^*$$

and

$$f^* \mathrm{Ev}_1 \xrightarrow{\epsilon_f} g^* \mathrm{Ev}_1 g_* g^* \xrightarrow{(\beta_f^{-1})} g^* g_* \mathrm{Ev}_1 g^* \xrightarrow{\eta_f} \mathrm{Ev}_1 f^*$$

Writing down the definition of the maps in the square, we see that we have to show the commutation of the outer square in the following diagram. When an arrow is obtained from another one by a clear functoriality, we omit the functor from the notation as well; for instance the first vertical

arrow in the top left should be named $g^* \text{Sus}^1 f_* w_S$).

$$\begin{array}{ccccc}
g^* \text{Sus}^1 f_* \mathbb{G}_m[1] & \xrightarrow{\sim} & g^* \text{Sus}^1 f_* \text{Ev}_1 \mathbb{Q} & \xrightarrow{\sim} & g^* \text{Sus}^1 \text{Ev}_1 f_* \mathbb{Q} & \xrightarrow{\eta} & g^* f_* \mathbb{Q} \\
\downarrow (\alpha_g) & & \downarrow (\alpha_g) & & \downarrow (\alpha_g) & & \downarrow \text{Ex}_*^* \\
\text{Sus}^1 g^* f_* \mathbb{G}_m[1] & \xrightarrow{\sim} & \text{Sus}^1 g^* f_* \text{Ev}_1 \mathbb{Q} & \xrightarrow{\sim} & \text{Sus}^1 g^* \text{Ev}_1 f_* \mathbb{Q} & \nearrow \eta & \\
\downarrow \text{Ex}_*^* & & \downarrow \text{Ex}_*^* & & (\gamma_g) \downarrow & & \\
\text{Sus}^1 f'_* g'^* \mathbb{G}_m[1] & \xrightarrow{\sim} & \text{Sus}^1 f'_* g'^* \text{Ev}_1 \mathbb{Q} & (*) & \text{Sus}^1 \text{Ev}_1 g^* f_* \mathbb{Q} & & \\
\downarrow R_{g'} & & \downarrow (\gamma_{g'}) & & \downarrow \text{Ex}_*^* & & \\
\text{Sus}^1 f'_* \mathbb{G}_m[1] & \xrightarrow{\sim} & \text{Sus}^1 f'_* \text{Ev}_1 \mathbb{Q} & \xrightarrow{\sim} & \text{Sus}^1 \text{Ev}_1 f'_* \mathbb{Q} & \xrightarrow{\eta} & f'_* \mathbb{Q}
\end{array}$$

The commutation of the three squares in the top left corner and of the bottom right corner follows directly by naturality of various natural transformations. The bottom left square commutes by Proposition 2.12. The top right square commutes by description 2.3 of (γ) .

It remains to show the commutation of $(*)$. By expanding (γ) using definition 2.3, we see that we have to show the commutativity of the outer square in the following diagram.

$$\begin{array}{ccccccc}
g^* f_* \text{Ev}_1 \mathbb{Q} & \xrightarrow{(\beta_f)} & & & g^* \text{Ev}_1 f_* \mathbb{Q} & & \\
\downarrow \text{Ex}_*^* & \searrow \epsilon_{g'} & & \swarrow \epsilon_{g'} & \downarrow \epsilon_g & & \\
f'_* g'^* \text{Ev}_1 \mathbb{Q} & & g^* f_* \text{Ev}_1 g' \mathbb{Q} & \xrightarrow{(\beta_f)} & g^* \text{Ev}_1 f_*(g')_* \mathbb{Q} & & g^* \text{Ev}_* g_* g'^* f_* \mathbb{Q} \\
\downarrow \epsilon_{g'} & \swarrow \text{Ex}_*^* & \downarrow (\beta_{g'}) & & \downarrow \sim & \swarrow \text{Ex}_*^* & \downarrow (\beta_g^{-1}) \\
f'_* g'^* \text{Ev}_1 g'_* \mathbb{Q} & & g^* f_* g'_* \text{Ev}_1 \mathbb{Q} & & g^* \text{Ev}_1 g_* f'_* \mathbb{Q} & & g^* g_* \text{Ev}_1 g^* f_* \mathbb{Q} \\
\downarrow (\beta_{g'}^{-1}) & \swarrow \text{Ex}_*^* & \downarrow \sim & & \downarrow (\beta_g^{-1}) & \swarrow \text{Ex}_*^* & \downarrow \eta_g \\
f'_* g'^* g'_* \text{Ev}_* \mathbb{Q} & & g^* g_* f'_* \text{Ev}_1 \mathbb{Q} & \xrightarrow{(\beta_{f'})} & g^* g_* \text{Ev}_1 f'_* \mathbb{Q} & & \text{Ev}_1 g^* f_* \mathbb{Q} \\
\downarrow \eta_{g'} & \swarrow \eta_g & & & \downarrow \eta_g & \swarrow \eta_g & \downarrow \text{Ex}_*^* \\
f'_* \text{Ev}_1 \mathbb{Q} & \xrightarrow{(\beta_{f'})} & & & \text{Ev}_1 f'_* \mathbb{Q} & &
\end{array}$$

The commutation of each of the subdiagrams follow from naturality properties of various natural transformations and from the definition of the exchange maps Ex_*^* . This completes the proof. \square

3. MOTIVIC PICARD FUNCTOR

We introduce and study the motivic Picard functor ω^1 , which is a (mixed motivic, relative) generalisation of the Picard variety of a smooth projective variety over a field. We also study in parallel the 0-motivic analogue ω^0 . Although some basic results on ω^0 from Section 3.1 are used in Sections 1 and 4, the main results are not used in the rest of the paper.

3.1. Definition and elementary properties.

Definition 3.1. Let $n \geq 0$. The full embedding $\iota^n : \mathbf{DA}^n(S) \hookrightarrow \mathbf{DA}^{\text{coh}}(S)$ preserves small sums, hence by Brown representability (see e.g. [Ayo07a, Proposition 2.1.21]) admits a right adjoint $\omega^n : \mathbf{DA}^{\text{coh}}(S) \rightarrow \mathbf{DA}^n(S)$. We also write ω^n for the functor $\mathbf{DA}^{\text{coh}}(S) \rightarrow \mathbf{DA}^{\text{coh}}(S)$ obtained by postcomposing with ι^n . We write $\delta^n : \omega^n \rightarrow \text{id}$ for the natural transformation induced by the counit.

Remark 3.2. The definition above can be extended to the whole of $\mathbf{DA}(S)$, but the resulting functors are not well-behaved; in particular, they do not respect compactness. Here is the simplest example of this phenomenon. Let k be an algebraically closed field. It is easy to see that the category $\mathbf{DA}_{0,c}(k)$ is equivalent to the bounded derived category of the category of finite dimensional

\mathbb{Q} -vector spaces. In particular Hom groups in this category are finite dimensional. On the other hand, $\mathbf{DA}(k)(\mathbb{Q}_k, \mathbb{Q}_k(1)[1]) \simeq k^\times \otimes \mathbb{Q}$ (Proposition B.6) is not finite dimensional in general. This shows $\omega^0(\mathbb{Q}(1))$ is not compact.

We start by giving some general formal properties of all the ω^n .

Proposition 3.3. *Let S be a noetherian finite-dimensional scheme.*

- (i) *Let $M \in \mathbf{DA}^n(S)$. Then we have an isomorphism $\delta^n(M) : \omega^n(M) \simeq M$ and the natural transformation $\delta^n(\omega^n) : \omega^n \circ \omega^n \rightarrow \omega^n$ is invertible.*
- (ii) *Let $f : T \rightarrow S$ be any morphism of schemes. There is a natural transformation $\alpha_f^n : f^* \omega^n \rightarrow \omega^n f^*$ making the triangles*

$$\begin{array}{ccc} f^* \omega^n & \xrightarrow{\alpha_f^n} & \omega^n f^* \\ & \searrow f^*(\delta^n) & \downarrow \delta^n(f^*) \\ & & f^* \end{array} \quad \text{and} \quad \begin{array}{ccc} \omega^n f^* \omega^n & \xrightarrow{\delta^n(f^* \omega^n)} & f^* \omega^n \\ & \searrow (\omega^n f^*)(\delta^n) & \downarrow \alpha_f^n \\ & & \omega^n f^* \end{array}$$

commutative.

- (iii) *Let $f : T \rightarrow S$ be any morphism of schemes. The natural transformation $\omega^n f_*(\delta^n)$ is invertible. Moreover there is a natural transformation $\beta_f^n : \omega^n f_* \rightarrow f_* \omega^n$ such that*
 - a) *the following triangles*

$$\begin{array}{ccc} \omega^n f_* & \xrightarrow{\beta_f^n} & f_* \omega^n \\ & \searrow \delta^n(f_*) & \downarrow f_*(\delta^n) \\ & & f_* \end{array} \quad \text{and} \quad \begin{array}{ccc} \omega^n f_* \omega^n & \xrightarrow{\omega^n(f_* \delta^n)} & \omega^n f_* \\ & \searrow \delta^n(f_* \omega^n) & \downarrow \beta_f^n \\ & & f_* \omega^n \end{array}$$

are commutative,

- b) $\omega^n(\beta_f^n)$ *is invertible for any f , and*
- c) β_f^n *is invertible for f finite.*
- (iv) *Let $e : T \rightarrow S$ be a quasi-finite morphism of schemes. There exists a natural transformation $\eta_e^n : e_! \omega^n \rightarrow \omega^n e_!$ such that*
 - a) *the following triangles*

$$\begin{array}{ccc} e_! \omega^n & \xrightarrow{\eta_e^n} & \omega^n e_! \\ & \searrow e_!(\delta^n) & \downarrow \delta^n(e_!) \\ & & e_! \end{array} \quad \text{and} \quad \begin{array}{ccc} \omega^n e_! \omega^n & \xrightarrow{\delta^n(e_! \omega^n)} & e_! \omega^n \\ & \searrow (\omega^n e_!)(\delta^n) & \downarrow \eta_e^n \\ & & \omega^n e_! \end{array}$$

commute, and

- b) *when e is finite, η_e^n is invertible and coincides with β_e^{-1} modulo the natural isomorphism $e_! \simeq e_*$.*
- (v) *Let $e : T \rightarrow S$ be a quasi-finite morphism. The natural transformation $\omega^n e^!(\delta^n)$ is invertible. Moreover there is a natural transformation $\gamma_e^n : \omega^n e^! \rightarrow e^! \omega^n$ such that*
 - a) *the following triangles*

$$\begin{array}{ccc} \omega^n e^! & \xrightarrow{\gamma_e^n} & e^! \omega^n \\ & \searrow \delta^n(e^!) & \downarrow e^!(\delta^n) \\ & & e^! \end{array} \quad \text{and} \quad \begin{array}{ccc} \omega^n e^! \omega^n & \xrightarrow{\omega^n(e^! \delta^n)} & \omega^n e^! \\ & \searrow \delta^n(e^! \omega^n) & \downarrow \gamma_e^n \\ & & e^! \omega^n \end{array}$$

are commutative,

- b) $\omega^n(\gamma_e^n)$ *is invertible for any e quasi-finite, and*
- c) γ_e^n *is invertible for e étale.*
- (vi) *Let $j : U \rightarrow S$ and $i : Z \rightarrow S$ be complementary open and closed immersions. Let $M \in \mathbf{DA}^{\text{coh}}(S)$ with $j^* M \in \mathbf{DA}^n(S)$. Then the morphism $i^* \omega^n M \rightarrow \omega^n i^* M$ is invertible.*

Remark 3.4. The formulation of Proposition 3.3 follows closely the one of [AZ12, Proposition 2.16] about ω^0 . More precisely, it is a direct generalization to all ω^n and to more general base schemes of all statements of loc. cit., minus the assertion in (ii) that α_f^0 is invertible for f smooth and the statement (vii) that ω^0 preserves compact objects. Unlike the others, these properties of ω^0 are not formal; in loc. cit., they follow from the key Proposition 2.11. We study their generalization to more general base schemes and higher n 's below.

Proof. We can apply verbatim the proof of [AZ12, Proposition 2.16] up to the sentence “To complete the proof (...)” on page 319. Notice that the rest of the proof after that sentence establishes the last assertion in (ii) together with (vii), which are precisely the points we are not claiming.

More precisely, up to that sentence, the proof of loc. cit. uses only general properties of \mathbf{DA} , the definition of ω^0 as adjoint, and the following permanence properties of cohomological 0-motives under the six operations.

- For all morphisms f , the functor f^* preserves \mathbf{DA}^0 .
- For all finite morphisms f , the functor f_* preserves \mathbf{DA}^0 .
- For all quasi-finite morphism e , the functor $e_!$ preserves \mathbf{DA}^0 .

The generalisation of these properties to \mathbf{DA}^n are established in the necessary generality in Proposition 1.16. \square

Here is another useful common properties of the ω^n 's.

Lemma 3.5. *Let S be a noetherian finite dimensional scheme and $n \in \mathbb{N}$. The functor $\omega^n : \mathbf{DA}^{\text{coh}}(S) \rightarrow \mathbf{DA}^n(S)$ commutes with small sums.*

Proof. The inclusion functor $\mathbf{DA}^n(S) \rightarrow \mathbf{DA}^{\text{coh}}(S)$ sends compact objects to compact objects by Lemma 1.8, hence by [Ayo07a, Lemme 2.1.28], its right adjoint ω^n commutes with infinite sums. \square

Proposition 3.6. *Let S be noetherian of finite dimension.*

- (i) *Let $f : X \rightarrow S$ be a smooth proper morphism of schemes. Let $X \xrightarrow{f^\circ} \pi_0(X/S) \xrightarrow{\pi_0(f)} S$ be its Stein factorisation, with $\pi_0(f)$ finite étale. Then there is a natural isomorphism*

$$\omega^0(f_*\mathbb{Q}_X) \xrightarrow{\sim} \pi_0(f)_*\mathbb{Q}_{\pi_0(X/S)}.$$

- (ii) *The functor ω^0 preserves geometrically smooth objects. More precisely, it sends $\mathbf{DA}_{\text{gsm}}^{\text{coh}}(S)$ to $\mathbf{DA}_{\text{gsm}}^0(S)$ and $\mathbf{DA}_{\text{gsm},c}^{\text{coh}}(S)$ to $\mathbf{DA}_{\text{gsm},c}^0(S)$. Moreover, for any $M \in \mathbf{DA}_{\text{gsm}}^{\text{coh}}(S)$ and any morphism $f : T \rightarrow S$, the natural morphism $\alpha_f^0(M) : f^*\omega^0 M \rightarrow \omega^0 f^* M$ is an isomorphism.*
- (iii) *The morphism α_f^0 is invertible for f smooth.*
- (iv) *The functor ω^0 preserves compact objects. More precisely, it sends $\mathbf{DA}_c^{\text{coh}}(S)$ to $\mathbf{DA}_c^0(S)$.*

Remark 3.7. These results were proved in [AZ12, §2] under the assumption that S is quasi-projective over a field k and f is projective.

Proof. It is easy to see from the definition of geometrically smooth motives and the fact that π_0 commutes with base change that point (ii) follows from (i). We now notice that the end of the proof of [AZ12, Proposition 2.16] (starting at “To complete the proof (...)”), which deduces (iii) and (iv) in the situation of loc. cit. from [AZ12, Proposition 2.11], applies verbatim and reduce Statements (ii)-(iv) to the sole Statement (i).

To prove Statement (i), it is enough by the Yoneda lemma to establish that for all $N \in \mathbf{DA}^0(S)$, the natural map $\pi_0(f)_*\mathbb{Q} \rightarrow f_*\mathbb{Q}_X$ induces an isomorphism

$$\mathbf{DA}(S)(N, \pi_0(f)_*\mathbb{Q}) \xrightarrow{\sim} \mathbf{DA}(S)(N, f_*\mathbb{Q}_X).$$

By Proposition 1.27, we have $\mathbf{DA}^0(S) = \mathbf{DA}_0(S)$. It is thus enough to show that for all $e : U \rightarrow S$ étale and $n \in \mathbb{Z}$, we have an isomorphism

$$\mathbf{DA}(S)(e_{\#}\mathbb{Q}_U[-n], \pi_0(f)_*\mathbb{Q}) \xrightarrow{\sim} \mathbf{DA}(S)(e_{\#}\mathbb{Q}_U[-n], f_*\mathbb{Q}_X).$$

By the $(e_{\#}, e^*)$ adjunction, proper base change, and the fact that π_0 commutes with smooth base change, we see that we can assume $e = \text{id}$. We are thus left to prove that for all $n \in \mathbb{Z}$, we have

$$\mathbf{DA}(\pi_0(X/S))(\mathbb{Q}, \mathbb{Q}[n]) \xrightarrow{\sim} \mathbf{DA}(X)(\mathbb{Q}, \mathbb{Q}[n])$$

where the morphism is induced by pullback by f° . The morphism f° is smooth proper with geometrically connected fibers, so this follows from Proposition B.5 (iv). \square

Here are some corollaries of Proposition 3.6.

Corollary 3.8. *Let S be a noetherian finite-dimensional scheme.*

- (i) *Let M be in $\mathbf{DA}_{\text{hom}}(S)$ and N be in $\mathbf{DA}^{\text{coh}}(S)$. Then the morphism $\delta^0(N)$ induces an isomorphism*

$$\mathbf{DA}(S)(M, \omega^0 N) \xrightarrow[\sim]{\delta^0(N)_*} \mathbf{DA}(S)(M, N).$$

- (ii) *We have $\mathbf{DA}_{\text{hom}}(S) \cap \mathbf{DA}^{\text{coh}}(S) = \mathbf{DA}^0(S)$.*
 (iii) *For all $N \in \mathbf{DA}^{\text{coh}}(S)$ we have $\omega^0(N(-1)) \simeq 0$.*
 (iv) *For all $N \in \mathbf{DA}^{\text{coh}}(S)$ and $d \geq 1$, we have*

$$\omega^1(N(-d)) \simeq \begin{cases} (\omega^0 N)(-1), & d = 1 \\ 0, & d \geq 2 \end{cases}.$$

Proof. We first prove (i). It is enough to show the isomorphism for a generator of $\mathbf{DA}_{\text{hom}}(S)$, namely $M = g_\# \mathbb{Q}_X[n]$ for $g : X \rightarrow S$ a smooth morphism and $n \in \mathbb{Z}$. By naturality of the adjunction which underlies δ^0 , we have a commutative square

$$\begin{array}{ccc} \mathbf{DA}(S)(g_\# \mathbb{Q}_X[n], \omega^0 N) & \xrightarrow{\delta^0(N)_*} & \mathbf{DA}(S)(g_\# \mathbb{Q}_X[n], N) \\ \sim \downarrow & & \downarrow \sim \\ \mathbf{DA}(X)(\mathbb{Q}_X[n], g^* \omega^0 N) & \xrightarrow{\delta^0(N)_*} & \mathbf{DA}(X)(\mathbb{Q}_X[n], g^* N). \end{array}$$

The first commutative triangle in Proposition 3.3 (ii) shows that we have a commutative square

$$\begin{array}{ccc} \mathbf{DA}(X)(\mathbb{Q}_X[n], g^* \omega^0 N) & \xrightarrow{\delta^0(N)_*} & \mathbf{DA}(X)(\mathbb{Q}_X[n], g^* N) \\ \alpha_g(N) \downarrow & & \parallel \\ \mathbf{DA}(X)(\mathbb{Q}_X[n], \omega^0 g^* N) & \xrightarrow{\delta^0(g^* N)_*} & \mathbf{DA}(X)(\mathbb{Q}_X[n], g^* N). \end{array}$$

Since g is smooth, the left vertical map is an isomorphism by Proposition 3.6 (ii); the bottom map is an isomorphism because $\mathbb{Q}_X[n]$ is a cohomological 0-motive. Putting this together with the previous commutative square concludes the proof of (i).

Statement (ii) follows directly from (i) applied to the identity map of an object in $\mathbf{DA}^{\text{coh}}(S) \cap \mathbf{DA}_{\text{hom}}(S)$.

To prove Statement (iii), we must show that for all $M \in \mathbf{DA}^0(S)$, we have $\mathbf{DA}(S)(M, N(-1)) = 0$. Since $\mathbf{DA}^0(S) = \mathbf{DA}_0(S)$ by Proposition 1.27 and $\mathbf{DA}_{\text{hom}}(S)$ is stable by positive twists by Proposition 1.10 (iv), the motive $M(1)$ is homological. By (i), this implies that $\mathbf{DA}(S)(M(1), N) \simeq \mathbf{DA}(S)(M(1), \omega^0 N)$. In other words, we can assume that both M and N are 0-motives. The statement to be proven is triangulated and commutes with infinite sums in M , so that we can assume that M is a generator of the form $e_\# \mathbb{Q}_U[n]$ for $e : U \rightarrow S$ an étale morphism and $n \in \mathbb{Z}$. Since this is a compact object, we can similarly assume that N is a generator of $\mathbf{DA}^0(S)$, of the form $f_* \mathbb{Q}_V[m]$ for $f : V \rightarrow S$ a finite morphism. We then have $\mathbf{DA}(S)(M, N(-1)) \simeq \mathbf{DA}(U \times_S V)(\mathbb{Q}, \mathbb{Q}(-1)[m - n])$. This group vanishes by Proposition B.2.

By (iii), we only need to establish (iv) in the case $d = 1$. The motive $\omega^0(N)(-1)$ is in $\mathbf{DA}^1(S)$ by Proposition 1.10 (ii). Hence by the Yoneda lemma, it is enough to show that for all $M \in \mathbf{DA}^1(S)$, the map $\delta^0(N)$ induces an isomorphism

$$\mathbf{DA}(S)(M, (\omega^0 N)(-1)) \xrightarrow[\sim]{\delta^0(N)_*} \mathbf{DA}(S)(M, N(-1)).$$

By Proposition 1.27, we have $\mathbf{DA}^1(S) = \mathbf{DA}_1(S)(-1)$. Write $M = M'(-1)$ with $M' \in \mathbf{DA}_1(S)$. In particular, M' is an homological motive. We have a commutative square

$$\begin{array}{ccc} \mathbf{DA}(S)(M, (\omega^0 N)(-1)) & \xrightarrow{\delta^0(N)_*} & \mathbf{DA}(S)(M, N(-1)) \\ \sim \downarrow & & \downarrow \sim \\ \mathbf{DA}(S)(M', \omega^0 N) & \xrightarrow{\delta^0(N)_*} & \mathbf{DA}(S)(M', N) \end{array}$$

The bottom map is an isomorphism by (i), and this concludes the proof in case $d = 1$. \square

We now compute ω^0 for some motives attached to commutative group schemes.

- Proposition 3.9.** (i) *Let G be an abelian scheme or a lattice over S ; then $\omega^0(\Sigma^\infty G_{\mathbb{Q}}(-1)) \simeq 0$.*
(ii) *Let T be a torus over S . Let $X_*(T)$ be the cocharacter lattice of T . Then $\Sigma^\infty T_{\mathbb{Q}}(-1) \simeq \Sigma^\infty X_*(T)_{\mathbb{Q}}$ is in $\mathbf{DA}_{0,c}(S)$.*
(iii) *Let $\mathbb{M} \in \mathcal{M}_1(S)$ and $W_{-2}\mathbb{M}$ be its toric part. Then $\omega^0(\Sigma^\infty \mathbb{M}(-1)) \simeq \Sigma^\infty X_*(W_{-2}\mathbb{M})_{\mathbb{Q}}$.*

Proof. First of all, we note that the objects to which we wish to apply ω^0 are in $\mathbf{DA}^1(S) \subset \mathbf{DA}^{\text{coh}}(S)$ by Corollary 2.13 and Proposition 1.27.

We first prove (i). Let G be an abelian scheme or a lattice and $M = \Sigma^\infty G_{\mathbb{Q}}$. The category $\mathbf{DA}^0(S) = \mathbf{DA}_0(S)$ is compactly generated by objects of the form $e_{\sharp}\mathbb{Q}_U$ for $e : U \rightarrow S$ étale. By adjunction and Proposition 2.2, we are reduced to showing that for all $n \in \mathbb{Z}$, we have $\mathbf{DA}(S)(\mathbb{Q}_S, M[n]) = 0$.

By using Zariski descent for $\mathbf{DA}(-)$ and Proposition 2.2, we can reduce to the case when $S = \mathbf{Spec}(R)$ is affine. Then by a continuity argument, we can reduce to the case where R is of finite type of a Dedekind ring, in particular satisfying resolution of singularities by alterations. Then S admits an h -hypercovering with regular terms. By using such an hypercovering, cohomological h -descent for $\mathbf{DA}(-)$ and Proposition 2.2, we reduce to the case where S is regular.

If G is a lattice, we can then in this case using Lemma A.2 write its motive as a direct factor $f_*\mathbb{Q}$ for f finite étale, and we are done by adjunction and Proposition B.2. If G is an abelian scheme, we know from [AHPL14, Theorem 3.3] (essentially, in this case, the theorem of Deninger and Murre on Chow motives of abelian schemes over a regular base) that the motive $\Sigma^\infty G_{\mathbb{Q}}$ is geometrically smooth, thus smooth, and compact. Because it is smooth, we can apply to it absolute purity in the form of Proposition 1.7. Let $i : Z \rightarrow S$ be a codimension d regular immersion and $j : U \rightarrow S$ its open complement. By colocalisation and Proposition 2.2, we get a long exact sequence

$$\dots \rightarrow \mathbf{DA}(Z)(\mathbb{Q}_Z, \Sigma^\infty G_Z(-d)[n-2d]) \rightarrow \mathbf{DA}(S)(\mathbb{Q}_S, \Sigma^\infty G[n]) \rightarrow \mathbf{DA}(U)(\mathbb{Q}_U, \Sigma^\infty G_U[n]) \rightarrow \dots$$

By the vanishing statements of Corollary 3.8, we then obtain that

$$\mathbf{DA}(S)(\mathbb{Q}_S, \Sigma^\infty G[n]) \simeq \mathbf{DA}(U)(\mathbb{Q}_U, \Sigma^\infty G_U[n])$$

for all $n \in \mathbb{Z}$. By stratifying a more general closed subset of S with regular strata, we see that the same result in fact holds for any dense open set U . By continuity for \mathbf{DA} , this implies that we can reduce to the case where S is the spectrum of a field k .

We can then write G as direct factor of the Jacobian of a smooth projective geometrically connected curve $f : C \rightarrow \mathbf{Spec}(k)$ with a rational point [Kat99, Theorem 11]. By Proposition 2.5 and relative purity, we have

$$\mathbb{Q}(-1)[-2] \oplus \Sigma^\infty \text{Jac}(C)_{\mathbb{Q}}(-1)[-2] \oplus \mathbb{Q} \simeq f_*\mathbb{Q}_C.$$

We have $\mathbf{DA}(k)(\mathbb{Q}_k, \mathbb{Q}_k(-1)[n]) = 0$ for all n (Proposition B.2). By adjunction, we have

$$\mathbf{DA}(k)(\mathbb{Q}_k, f_*\mathbb{Q}_C[n]) \simeq \mathbf{DA}(C)(\mathbb{Q}_C, \mathbb{Q}_C[n])$$

which is isomorphic to \mathbb{Q} for $n = 0$ and 0 otherwise (Proposition B.5). Similarly, we have $\mathbf{DA}(k)(\mathbb{Q}_k, \mathbb{Q}_k[n])$ is isomorphic to \mathbb{Q} for $n = 0$ and 0 otherwise. Putting everything together, for any n we deduce that $\mathbf{DA}(k)(\mathbb{Q}_k, \Sigma^\infty \text{Jac}(C)_k[n]) = 0$, as required.

We prove (ii). Let T be a torus. We have $\Sigma^\infty T_{\mathbb{Q}}(-1) \simeq \Sigma^\infty X_*(T)_{\mathbb{Q}}$ by Corollary 2.9. The motive $\Sigma^\infty X_*(T)_{\mathbb{Q}}$ lies in $\mathbf{DA}_0(S)$: this can be tested pointwise by Proposition 1.23, and over

a field a lattice is a direct factor of the motive of a finite étale morphism by Lemma A.2. This concludes the proof.

Finally, (iii) follows immediately from the two previous points by devissage of a Deligne 1-motive along its weight filtration. \square

Corollary 3.10. *Assume S regular. Let $f : X \rightarrow S$ be a smooth projective Pic-smooth morphism of schemes. Then there is an isomorphism*

$$\omega_0(\Sigma^\infty \mathbf{P}(X/S)_{\mathbb{Q}}(-1)[-2]) \simeq \pi_0(f)_* \mathbb{Q}$$

Proof. First, by Corollary 2.33, Proposition 2.11 and Proposition 1.27, the motive $\Sigma^\infty \mathbf{P}(X/S)_{\mathbb{Q}}(-1)[-2]$ is in $\mathbf{DA}^1(S)$, and it makes sense to apply ω^0 . More precisely, Corollary 2.33 together with Proposition 3.9 shows that there is an isomorphism

$$\omega_0(\Sigma^\infty \mathbf{P}(X/S)_{\mathbb{Q}}(-1)[-2]) \simeq \Sigma^\infty X_*(\mathrm{Res}_{\pi_0(f)} \mathbb{G}_m)_{\mathbb{Q}}.$$

The cocharacter lattice of the Weil restriction $\mathrm{Res}_{\pi_0(f)} \mathbb{G}_m$ is the permutation lattice associated to $\pi_0(f)$; hence, $\Sigma^\infty X_*(\mathrm{Res}_{\pi_0(f)} \mathbb{G}_m)_{\mathbb{Q}} \simeq \pi_0(f)_* \mathbb{Q}$ as required. \square

3.2. The functors ω^n over a perfect field. In this short section, we explain how, for S the spectrum of a perfect field k , the functors ω^0 and ω^1 are related to the functors $L\pi_0$ and $L\mathrm{Alb}$ studied in [BVK] and [ABV09].

We need to connect our setup with the categories of effective motives with transfers over k . First, we define for every $n \in \mathbb{N}$ the category $\mathbf{DM}_{n,(c)}^{\mathrm{eff}}(k)$ in a similar as $\mathbf{DA}_{n,(c)}^{\mathrm{eff}}(k)$, replacing $\mathbf{DA}(k)$ with $\mathbf{DM}^{\mathrm{eff}}(k)$ and $f_{\#} \mathbb{Q}_X$ with $M_k^{(\mathrm{eff}), \mathrm{tr}}(X)$ for $f : X \rightarrow \mathbf{Spec}(k)$ smooth. We also define the category $\mathbf{DM}_{\mathrm{hom},(c)}(k)$ (resp. $\mathbf{DM}_{(c)}^{\mathrm{coh}}(k)$) in a similar way as $\mathbf{DA}_{\mathrm{hom},(c)}(k)$ (resp. $\mathbf{DA}_{(c)}^{\mathrm{coh}}(k)$).

Recall that by construction of $\mathbf{DM}(k)$ as homotopy category of spectra, there is an adjunction

$$\Sigma_{\mathrm{tr}}^\infty : \mathbf{DM}^{\mathrm{eff}}(k) \rightleftarrows \mathbf{DM}(k) : \Omega_{\mathrm{tr}}^\infty.$$

Lemma 3.11. *Let k be a perfect field and $n \in \mathbb{N}$. The adjoint pairs $\Sigma_{\mathrm{tr}}^\infty \dashv \Omega_{\mathrm{tr}}^\infty$ and $a_{\mathrm{tr}} \dashv o^{\mathrm{tr}}$ restrict to equivalences of categories*

$$\begin{aligned} \mathbf{DA}_{n,(c)}(k) &\xrightleftharpoons[o^{\mathrm{tr}}]{a_{\mathrm{tr}}} \mathbf{DM}_{n,(c)}(k) \xrightleftharpoons[\Sigma^\infty]{\Omega^\infty} \mathbf{DM}_{n,(c)}^{\mathrm{eff}}(k), \\ \mathbf{DA}_{\mathrm{hom},(c)}(k) &\xrightleftharpoons[o^{\mathrm{tr}}]{a_{\mathrm{tr}}} \mathbf{DM}_{\mathrm{hom},(c)}(k) \xrightleftharpoons[\Sigma^\infty]{\Omega^\infty} \mathbf{DM}_{(c)}^{\mathrm{eff}}(k) \\ \mathbf{DA}_{(c)}^{\mathrm{coh}}(k) &\xrightleftharpoons[o^{\mathrm{tr}}]{a_{\mathrm{tr}}} \mathbf{DM}_{(c)}^{\mathrm{coh}}(k) \\ \mathbf{DA}_{(c)}^n(k) &\xrightleftharpoons[o^{\mathrm{tr}}]{a_{\mathrm{tr}}} \mathbf{DM}_{(c)}^n(k) \end{aligned}$$

Proof. The argument is essentially the same for the four series of equivalences; we only give the details for the first one. Recall that $a_{\mathrm{tr}} : \mathbf{DM}(S) \rightleftarrows \mathbf{DA}(S)$ is an equivalence of categories for all S geometrically unibranch [CD, Corollary 16.2.22], hence in particular for all X smooth over k . By construction of a_{tr} , we have $a_{\mathrm{tr}} f_{\#} \simeq f_{\#} a_{\mathrm{tr}}$ for f smooth.

Moreover, the functors a_{tr} , o^{tr} commute with small sums and preserve compact objects; indeed, a_{tr} is a left adjoint, it preserves constructible objects by construction ($a_{\mathrm{tr}} M_k(X) \simeq M_k^{\mathrm{tr}}(X)$) and compact = constructible holds in $\mathbf{DM}(k)$ as well as in $\mathbf{DA}(k)$, while o_{tr} has a left adjoint which preserves compact objects hence it commutes with small sums, and it preserves constructible objects because $o^{\mathrm{tr}} M_k^{\mathrm{tr}}(X) \simeq o^{\mathrm{tr}} a_{\mathrm{tr}} M_k(X) \simeq M_k(X)$ since we have an equivalence of categories. Put together, these facts imply that $a_{\mathrm{tr}} \dashv o^{\mathrm{tr}}$ restricts to an equivalence of categories between $\mathbf{DA}_{n,(c)}$ and $\mathbf{DM}_{n,(c)}$.

By Voevodsky's cancellation theorem [Voe10], the functor $\Sigma_{\mathrm{tr}}^\infty : \mathbf{DM}^{\mathrm{eff}} \rightarrow \mathbf{DM}(k)$ is fully faithful. We have $\Sigma_{\mathrm{tr}}^\infty f_{\#} \simeq f_{\#} \Sigma_{\mathrm{tr}}^\infty$ for $f : X \rightarrow \mathbf{Spec}(k)$ smooth; this shows that all subcategories $\mathbf{DM}_n(k)$ lies in the essential image of $\mathbf{DM}_{n,(c)}^{\mathrm{eff}}(k)$. Since $\Sigma_{\mathrm{tr}}^\infty$ and $\Omega_{\mathrm{tr}}^\infty$ both commute with small sums and preserve compact objects (this follows from the same line of argument as for $a_{\mathrm{tr}}, o^{\mathrm{tr}}$ above), this implies further that $\Sigma_{\mathrm{tr}}^\infty \dashv \Omega_{\mathrm{tr}}^\infty$ restricts to an equivalence of categories between $\mathbf{DM}_{n,(c)}(k)$ and $\mathbf{DM}_{n,(c)}^{\mathrm{eff}}(k)$. This completes the proof. \square

By [ABV09, Theorem 2.4.1] specialized to the case of \mathbb{Q} -coefficients, we have a functor

$$L\pi_0 : \mathbf{DM}^{\text{eff}}(k) \rightarrow \mathbf{DM}_0^{\text{eff}}(k)$$

(resp.

$$\text{LAlb} : \mathbf{DM}^{\text{eff}}(k) \rightarrow \mathbf{DM}_1^{\text{eff}}(k))$$

which is a left adjoint to the inclusion $\mathbf{DM}_0^{\text{eff}}(k) \rightarrow \mathbf{DM}^{\text{eff}}(k)$ (resp. $\mathbf{DM}_1^{\text{eff}}(k) \rightarrow \mathbf{DM}^{\text{eff}}(k)$) and restricts by [ABV09, Proposition 2.3.3] (resp. [ABV09, Proposition 2.4.7]) to a functor on compact objects

$$L\pi_0 : \mathbf{DM}_c^{\text{eff}}(k) \rightarrow \mathbf{DM}_{0,c}^{\text{eff}}(k)$$

(resp.

$$\text{LAlb} : \mathbf{DM}_c^{\text{eff}}(k) \rightarrow \mathbf{DM}_{1,c}^{\text{eff}}(k)).$$

More precisely, the functor $L\pi_0$ (resp. LAlb) in loc. cit. has as target category $D(\mathbf{HI}_{\leq 0}(k))$ (resp. $D(\mathbf{HI}_{\leq 1}(k))$), the derived category of the abelian category $\mathbf{Sh}(\mathbf{Spec}(k)_{\text{ét}}, \mathbb{Q})$ (resp. $\mathbf{HI}_{\leq 1}(k)$) of 1-motivic sheaves [ABV09, Definition 1.1.20]), which is equivalent by [ABV09, Lemma 2.3.1] (resp. [ABV09, Theorem 2.4.1.(i)]) to $\mathbf{DM}_0^{\text{eff}}(k)$ (resp. $\mathbf{DM}_1^{\text{eff}}(k)$), and the functor we call $L\pi_0$ (resp. LAlb) is obtained by composing the functor of loc. cit. by this equivalence.

By Proposition 1.25, the duality functor \mathbb{D}_k restricts to anti-equivalences of categories $\mathbf{DA}_c^{\text{coh}}(k)^{\text{op}} \simeq \mathbf{DA}_{\text{hom},c}(k)$ and $\mathbf{DA}_c^n(k)^{\text{op}} \simeq \mathbf{DA}_{n,c}(k)$ for any $n \in \mathbb{N}$. By Lemma 3.11, this implies that the inclusion $\mathbf{DA}_{0,c}(k) \rightarrow \mathbf{DA}_c(k)$ (resp. $\mathbf{DA}_{1,c}(k) \rightarrow \mathbf{DA}_c(k)$) admits as right adjoint the composition

$$\mathbb{D}_k \circ {}^{\text{tr}}\Sigma_{\text{tr}}^{\infty} L\pi_0 \Omega_{\text{tr}}^{\infty} a_{\text{tr}} \mathbb{D}_k : \mathbf{DA}_c^{\text{coh}}(k) \rightarrow \mathbf{DA}_c^0(k)$$

(resp.

$$\mathbb{D}_k \circ {}^{\text{tr}}\Sigma_{\text{tr}}^{\infty} \text{LAlb} \Omega_{\text{tr}}^{\infty} a_{\text{tr}} \mathbb{D}_k : \mathbf{DA}_c^{\text{coh}}(k) \rightarrow \mathbf{DA}_c^1(k)).$$

In the case $n = 0$, we already know that the functor ω^0 restricts to compact objects by Proposition 3.6 (iv), so that this right adjoint and the restriction of ω^0 (which we also denote by ω^0) coincide. In the case $n = 1$, this is also the case by the following observation. Write temporarily $\tilde{\omega}^1 := \mathbb{D}_k \circ {}^{\text{tr}}\Sigma_{\text{tr}}^{\infty} \text{LAlb} \Omega_{\text{tr}}^{\infty} a_{\text{tr}} \mathbb{D}_k$. Let $M \in \mathbf{DA}_c^{\text{coh}}(k)$. There is a morphism $\tilde{\omega}^1 M \rightarrow M$ in $\mathbf{DA}_c^{\text{coh}}(k)$, which by the adjunction property of ω^1 factors through a morphism $\tilde{\omega}^1 M \rightarrow \omega^1 M$ in $\mathbf{DA}^1(k)$. The category $\mathbf{DA}^1(k)$ is compactly generated, hence to show that this morphism is an isomorphism, it is enough to show that for every $N \in \mathbf{DA}_c^1(k)$, the induced morphism $\mathbf{DA}_c^1(k)(N, \tilde{\omega}^1 M) \rightarrow \mathbf{DA}_c^1(k)(N, \omega^1 M)$ is an isomorphism; but this follows from the adjunction properties of both functors. We deduce that ω^1 restricts to compact objects, and that its restriction is related to LAlb by the formula above. Let us summarize all those results for future reference.

Proposition 3.12. *Let k be a perfect field. The functors ω^0 and ω^1 restrict to compact objects. Moreover, when restricting to compact objects, we have isomorphisms of functors*

$$\omega^0 \simeq \mathbb{D}_k \circ {}^{\text{tr}}\Sigma_{\text{tr}}^{\infty} L\pi_0 \Omega_{\text{tr}}^{\infty} a_{\text{tr}} \mathbb{D}_k : \mathbf{DA}_c^{\text{coh}}(k) \rightarrow \mathbf{DA}_c^0(k)$$

and

$$\omega^1 \simeq \mathbb{D}_k \circ {}^{\text{tr}}\Sigma_{\text{tr}}^{\infty} \text{LAlb} \Omega_{\text{tr}}^{\infty} a_{\text{tr}} \mathbb{D}_k : \mathbf{DA}_c^{\text{coh}}(k) \rightarrow \mathbf{DA}_c^1(k).$$

Finally, we use another result of [ABV09] to show that the ω^n 's for $n \geq 2$ are not well-behaved, at least over “large” fields.

Proposition 3.13. *Let $n \geq 2$ and k be an algebraically closed field of infinite transcendence degree over \mathbb{Q} , e.g. $k = \mathbb{C}$. Then $\omega^n : \mathbf{DA}^{\text{coh}}(k) \rightarrow \mathbf{DA}^n(k)$ does not preserve compact objects.*

Proof. We prove this by contradiction. Assume that ω^n preserves compact object and write again $\omega^n : \mathbf{DA}_c^{\text{coh}}(k) \rightarrow \mathbf{DA}_c^n(k)$ for the restriction. By Proposition 1.25, the duality functor \mathbb{D}_k restricts to anti-equivalences of categories $\mathbf{DA}_c^{\text{coh}}(k)^{\text{op}} \simeq \mathbf{DA}_{\text{hom},c}(k)$ and $\mathbf{DA}_c^n(k)^{\text{op}} \simeq \mathbf{DA}_{n,c}(k)$. This implies that the composition $\mathbb{D}_k \circ (\omega^n)^{\text{op}} \circ \mathbb{D}_k : \mathbf{DA}_{\text{hom},c}(k) \rightarrow \mathbf{DA}_{n,c}(k)$ provides a left adjoint to the inclusion $\mathbf{DA}_{n,c}(k) \rightarrow \mathbf{DA}_{\text{hom},c}(k)$.

By Lemma 3.11, this also provides a left adjoint to $\mathbf{DM}_{n,c}^{\text{eff}}(k) \rightarrow \mathbf{DM}_c^{\text{eff}}(k)$. This contradicts [ABV09, §2.5] (note that the assumption there is the existence of a left adjoint to $\mathbf{DM}_n^{\text{eff}}(k) \rightarrow \mathbf{DM}^{\text{eff}}(k)$ but the proof only uses the existence of the adjoint on compact objects). \square

3.3. Computation and finiteness of ω^1 . We can now compute ω^1 in an important special case.

Theorem 3.14. *Let $f : X \rightarrow S$ be a smooth projective Pic-smooth morphism with S regular. The morphism $\Theta_f : \Sigma^\infty \mathbf{P}(X/S)(-1)[-2] \rightarrow f_* \mathbb{Q}_X$ of Section 2.3 induces an isomorphism*

$$\omega^1 f_* \mathbb{Q}_X \simeq \Sigma^\infty \mathbf{P}(X/S)(-1)[-2].$$

In particular, under these hypotheses, $\omega^1 f_ \mathbb{Q}_X$ is compact.*

Proof. First of all, the motive $\Sigma^\infty \mathbf{P}(X/S)$ lies in $\mathbf{DA}_{1,c}(S)$ by Corollary 2.33. By Proposition 1.27, this implies that $\Sigma^\infty (\mathbf{P}(X/S) \otimes \mathbb{Q})(-1)[-2]$ lies in $\mathbf{DA}_c^1(S)$. This implies that Θ_f induces a morphism $\Sigma^\infty (\mathbf{P}(X/S) \otimes \mathbb{Q})(-1)[-2] \rightarrow \omega^1 f_* \mathbb{Q}_X$; the claim is that this is an isomorphism. We just observed that $(\Sigma^\infty \mathbf{P}(X/S))(-1)[-2]$ is compact, so this will also establish the compactness claim.

We first treat the case when S is the spectrum of a field k . Let k^{perf} be a perfect closure of k and $h : \mathbf{Spec}(k^{\text{perf}}) \rightarrow \mathbf{Spec}(k)$ be the canonical morphism. By Proposition 2.38 and applying ω^1 , we have a commutative diagram

$$\begin{array}{ccc} h^* \Sigma^\infty \mathbf{P}(X/S)(-1)[-2] & \longrightarrow & \omega^1(h^* f_* \mathbb{Q}_X) \\ \downarrow v_h \circ R_h & & \downarrow \omega^1(\text{Ex}_*) \\ \Sigma^\infty \mathbf{P}(X_T/T)(-1)[-2] & \xrightarrow{\Theta_{f'}} & \omega^1(f'_* \mathbb{Q}_{X_T}). \end{array}$$

The morphism h is not smooth and we cannot directly apply Lemma 2.21. However, since f is Pic-smooth, the morphism $v_h : h^* P(X/k) \rightarrow P(X_{k^{\text{perf}}}/k^{\text{perf}})$ is an isomorphism if and only if the natural morphism $h^* \mathcal{NS}(X/k) \rightarrow \mathcal{NS}(X_{k^{\text{perf}}}/k^{\text{perf}})$ is. Let k^s be a separable closure of k and $\bar{k} = k^s k^{\text{perf}}$. Looking at the proof of Proposition 2.31, we find that $\mathcal{NS}(X/k)$ is represented by the $\text{Gal}(k^s/k)$ -module $\text{NS}(X_{k^s}/k^s)$ while $\mathcal{NS}(X_{k^{\text{perf}}}/k^{\text{perf}})$ is represented by the $\text{Gal}(\bar{k}/k^{\text{perf}})$ -module $\text{NS}(X_{\bar{k}}/\bar{k})$. Those two groups are canonically isomorphic, and we conclude that v_h is an isomorphism. Since R_h is an isomorphism, we see that the left vertical map in the diagram is an isomorphism. Moreover, since h is finite and purely inseparable, by the separation property of \mathbf{DA} and Lemma 1.18 (ii), we see that the natural morphism $\alpha_h : h^* \omega^1 \rightarrow \omega^1 f^*$ is an isomorphism. Together with the commutative diagram above, this shows that we can reduce the question of whether Θ_f is an isomorphism to the case of a perfect field.

Let us assume that k is perfect. By Proposition 1.27, the category $\mathbf{DA}^1(k)$ is compactly generated by motives of the form $g_\# \mathbb{Q}_C(-1)$ for a smooth curve $g : C \rightarrow k$. We thus have to show that for all such g and all $n \in \mathbb{Z}$, the map

$$\mathbf{DA}(k)(g_\# \mathbb{Q}_C(-1)[-n], \Sigma^\infty \mathbf{P}(X/k)(-1)[-2]) \xrightarrow{(\Theta_f)^*} \mathbf{DA}(k)(g_\# \mathbb{Q}_C(-1)[-n], f_* \mathbb{Q}_X)$$

induced by Θ_f is an isomorphism (this turns out that to hold for any smooth C , not only for curves, as the argument below shows).

First, using that $a^{\text{tr}} \Theta_f = \Theta_f^{\text{tr}}$ (modulo isomorphisms, Proposition 2.37), this is equivalent to asking whether the morphism

$$\mathbf{DM}(k)(g_\# \mathbb{Q}_C^{\text{tr}}, \Sigma_{\text{tr}}^\infty \mathbf{P}^{\text{tr}}(X/k)[n-2]) \xrightarrow{(\Theta_f^{\text{tr}})^*} \mathbf{DM}(k)(g_\# \mathbb{Q}_C^{\text{tr}}, f_* \mathbb{Q}_X^{\text{tr}}(1)[n])$$

is an isomorphism. Because $\mathbf{DM}^{(\text{eff})}(-)$ has internal homs and f is smooth, one has “projection formulas” isomorphisms

$$\Lambda = \Lambda_{M,N} : \mathbf{DM}^{(\text{eff})}(k)(M \otimes f_\# \mathbb{Q}, N) \simeq \mathbf{DM}^{(\text{eff})}(k)(M, f_* N)$$

natural in $M \in \mathbf{DM}^{(\text{eff})}(k), N \in \mathbf{DM}^{(\text{eff})}(X)$. Using the alternative description of Θ_f^{tr} from Proposition 2.37 with $u_X^{\text{eff}, \text{tr}}$ and the fact that $u_X^{\text{eff}, \text{tr}}$ is an isomorphism, we see that we have to

show that the top morphism in the diagram below is an isomorphism.

$$\begin{array}{ccc}
\mathbf{DM}(k)(g_{\#}\mathbb{Q}_C^{\mathrm{tr}}, \Sigma^{\infty} f_* \mathbb{Q}_X^{\mathrm{tr}}(1)[n]) & \longrightarrow & \mathbf{DM}(k)(g_{\#}\mathbb{Q}_C^{\mathrm{tr}}, f_* \mathbb{Q}^{\mathrm{tr}}(1)[n]) \\
\uparrow \Sigma^{\infty} \sim & & \uparrow \Lambda \sim \\
\mathbf{DM}^{\mathrm{eff}}(k)(g_{\#}\mathbb{Q}_C, f_* \mathbb{Q}_X^{\mathrm{tr}}(1)[n]) & & \mathbf{DM}(k)(g_{\#}\mathbb{Q}_C^{\mathrm{tr}} \otimes f_{\#}\mathbb{Q}, \mathbb{Q}^{\mathrm{tr}}(1)[n]) \\
\uparrow \Lambda \sim & & \uparrow \Sigma^{\infty} \sim \\
\mathbf{DM}^{\mathrm{eff}}(k)(g_{\#}\mathbb{Q}_C^{\mathrm{tr}} \otimes f_{\#}\mathbb{Q}_X^{\mathrm{tr}}, \mathbb{Q}^{\mathrm{tr}}(1)[n]) & \xlongequal{\quad} & \mathbf{DM}^{\mathrm{eff}}(g_{\#}\mathbb{Q}_C^{\mathrm{tr}} \otimes f_{\#}\mathbb{Q}_X^{\mathrm{tr}}, \mathbb{Q}^{\mathrm{tr}}(1)[n]).
\end{array}$$

This diagram commutes by an easy compatibility argument between Σ^{∞} and Λ . The maps induced by Σ^{∞} are isomorphisms because of the Cancellation theorem [Voe10] (this is where we use the hypothesis k perfect).

We now do the general case. We can assume S is connected, and so integral. The statement of the theorem is equivalent to the following: for all $M \in \mathbf{DA}_1(S)$, the map Θ_f induces an isomorphism

$$\mathbf{DA}(S)(M, (\Sigma^{\infty} \mathbf{P}(X/S))(-1)[-2]) \xrightarrow{\sim} \mathbf{DA}(S)(M, f_* \mathbb{Q}_X).$$

We first make a series of reformulations of this statement. By Proposition 1.27 and the definition of $\mathbf{DA}_1(S)$, the category $\mathbf{DA}^1(S)$ is compactly generated by objects of the form $g_{\#}\mathbb{Q}_C(-1)$ for a smooth curve $g : C \rightarrow S$. We can thus reformulate the theorem as follows: for every smooth curve $g : C \rightarrow S$ and all $n \in \mathbb{Z}$, the map

$$\mathbf{DA}(S)(g_{\#}\mathbb{Q}_C(-1)[-n], \Sigma^{\infty} \mathbf{P}(X/S)(-1)[-2]) \xrightarrow{\Theta_f^*} \mathbf{DA}(S)(g_{\#}\mathbb{Q}_C(-1)[-n], f_* \mathbb{Q}_X)$$

induced by Θ_f is an isomorphism. By adjunction, this is equivalently to the statement that the map

$$\mathbf{DA}(C)(\mathbb{Q}_C(-1)[-n], g^* \Sigma^{\infty} \mathbf{P}(X/S))(-1)[-2] \xrightarrow{(g^* \Theta_f)^*} \mathbf{DA}(C)(\mathbb{Q}_C(-1)[-n], g^* f_* \mathbb{Q}_X)$$

induced by $g^* \Theta_f$ is an isomorphism. Let $f' : X_C \rightarrow C$ be the pullback of f along g . The morphism f' is Pic-smooth by Remark and C is regular. By Proposition 2.38 and the fact that ν_g is an isomorphism because g is smooth (Lemma 2.21), the morphism $(g^* \Theta_f)_*$ above is an isomorphism if and only if the morphism

$$\mathbf{DA}(C)(\mathbb{Q}_C, (\Sigma^{\infty} \mathbf{P}(X_C/C))[n-2]) \xrightarrow{\Theta_{f'}^*} \mathbf{DA}(C)(\mathbb{Q}_C, f'_* \mathbb{Q}_{X_C}(1)[n])$$

is an isomorphism. In other words, since f' still satisfies all the hypotheses of the theorem, we can assume that $g = \mathrm{id}$.

By adjunction, the right-hand side is isomorphic to the motivic cohomology group $H_{\mathcal{M}}^{n,1}(X)$. Because S is regular, we know from Proposition B.6 how to compute it: it is zero for $\neq 1, 2$, and we have explicit morphisms relating it to $\mathcal{O}^{\times}(X)_{\mathbb{Q}}$ if $n = 1$ (resp. $\mathrm{Pic}(X)_{\mathbb{Q}}$ if $n = 2$). The idea of the rest of the proof is to apply a similar localisation argument to the proof of Proposition B.6 to the group

$$\mathrm{HP}^{n-2}(X/S) := \mathbf{DA}(S)(\mathbb{Q}_S, (\Sigma^{\infty} \mathbf{P}(X/S))[n-2]).$$

Let $j : U \rightarrow S$ be a non-empty open set and $i : Z \rightarrow S$ its reduced closed complement. Then by applying colocalisation, we get a commutative diagram

$$\begin{array}{ccccccc}
\cdots \rightarrow \mathbf{DA}(Z)(\mathbb{Q}_Z, i^! \Sigma^{\infty} \mathbf{P}(X/S)[n-2]) & \rightarrow & \mathrm{HP}^{n-2}(X/S) & \rightarrow & \mathrm{HP}^{n-2}(X_U/U) & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\cdots \rightarrow \mathbf{DA}(Z)(\mathbb{Q}_Z, i^!(f_* \mathbb{Q}_X(1)[n])) & \rightarrow & H_{\mathcal{M}}^{n,1}(X) & \rightarrow & H_{\mathcal{M}}^{n,1}(X_U) & \rightarrow & \cdots
\end{array}$$

As in the proof of Proposition B.6, we stratify $Z = Z_0 \subset Z_1 \subset \cdots \subset Z_d = \emptyset$ in such a way that for all k , the scheme $(Z_k \setminus Z_{k+1})_{\mathrm{red}}$ is regular of codimension d_k in S and in such a way that $(Z \setminus Z_1)$ contains all points of codimension 1 of Z in S (so that $d_k \geq 2$ for $k \geq 1$). Let $i_k : (Z_k \setminus Z_{k+1})_{\mathrm{red}} \rightarrow S$ be the corresponding regular locally closed immersion.

By Corollary 2.33, the motive $\Sigma^{\infty} \mathbf{P}(X/S)(-1)$ is in $\mathbf{DA}_{\mathrm{gsm}}^1(S)$. By absolute purity in the form of Proposition 1.7, for any k , we have $i_k^! \Sigma^{\infty} \mathbf{P}(X/S) \simeq i_k^* \mathbf{P}(X/S)(-d_k)[-2d_k]$. In particular,

by Corollary 3.8 (iii), we have $\omega^0(i_k^! \Sigma^\infty P(X/S)) \simeq 0$ for $k \geq 2$. This shows that by applying inductively absolute purity and colocalisation, we get a commutative diagram

$$\begin{array}{ccccccc} \dots \rightarrow \mathbf{DA}(Z)(\mathbb{Q}_{Z \setminus Z_1}, i_1^* \Sigma^\infty P(X/S)(-1)[n-4]) & \rightarrow & \mathrm{HP}^{n-2}(X/S) & \rightarrow & \mathrm{HP}^{n-2}(X_U/U) & \rightarrow & \dots \\ & & \downarrow & & \downarrow & & \\ \dots \longrightarrow & H^{n-2,0}(X_{Z \setminus Z_1}) & \longrightarrow & H_{\mathcal{M}}^{n,1}(X) & \longrightarrow & H_{\mathcal{M}}^{n,1}(X_U) & \longrightarrow \dots \end{array}$$

As in the proof of Proposition B.6, we make the notational abuse of replacing $Z \setminus Z_1$ by Z in the rest of the proof, since everything happens in codimension 1. The motive $i^* \Sigma^\infty P(X/S)(-1)[n-4]$ lies in $\mathbf{DA}^{\mathrm{coh}}(Z)$, so that

$$\mathbf{DA}(Z)(\mathbb{Q}_Z, i^* \Sigma^\infty P(X/S)(-1)[n-4]) \simeq \mathbf{DA}(Z)(\mathbb{Q}_Z, \omega^0(i^* \Sigma^\infty P(X/S)(-1)[n-4]))$$

Using Corollary 2.33, we apply Proposition 3.6 (ii) to get an isomorphism

$$\omega^0(i^* \Sigma^\infty P(X/S)(-1)[n-4]) \simeq i^* \omega^0(\Sigma^\infty P(X/S)(-1)[n-4]).$$

By Corollary 3.10, we then have

$$\omega^0(\Sigma^\infty P(X/S)(-1)[n-4]) \simeq \pi_0(f)_* \mathbb{Q}[n-2].$$

We deduce that

$$\mathbf{DA}(Z)(\mathbb{Q}_Z, i^! \Sigma^\infty P(X/S)[n-2]) \simeq H^{n-2,0}(\pi_0(X_Z/Z))$$

We rewrite this into the previous commutative diagram to get

$$\begin{array}{ccccccc} \dots \longrightarrow & H^{n-2,0}(\pi_0(X_Z/Z)) & \longrightarrow & \mathrm{HP}^{n-2}(X/S) & \longrightarrow & \mathrm{HP}^{n-2}(X_U/U) & \longrightarrow \dots \\ & \downarrow (\pi_0)^* & & \downarrow & & \downarrow & \\ \dots \longrightarrow & H^{n-2,0}(X_Z) & \longrightarrow & H_{\mathcal{M}}^{n,1}(X) & \longrightarrow & H_{\mathcal{M}}^{n,1}(X_U) & \longrightarrow \dots \end{array}$$

By Proposition B.5, since X_Z and $\pi_0(X_Z/Z)$ are both regular and have the same set of connected components, the map $(\pi_0)^*$ is an isomorphism for all n , and the groups $H^{n-2,0}(X_Z)$ vanish for $\neq 2$. As a consequence, we see that the pullback map $\mathrm{HP}^{n-2}(S) \rightarrow \mathrm{HP}^{n-2}(U)$ is an isomorphism for $\neq 1, 2$, and there is a commutative diagram

$$\begin{array}{ccccccccc} 0 \rightarrow \mathrm{HP}^{-1}(X/S) \rightarrow \mathrm{HP}^{-1}(X_U/U) \rightarrow \mathbb{Q}^{\pi_0(X_Z)} \rightarrow \mathrm{HP}^0(X/S) \rightarrow \mathrm{HP}^0(X_U/U) \rightarrow 0 \\ \downarrow \quad \quad \downarrow \quad \quad \parallel \quad \quad \downarrow \quad \quad \downarrow \\ 0 \rightarrow H_{\mathcal{M}}^{1,1}(X_S) \rightarrow H_{\mathcal{M}}^{1,1}(X_U) \rightarrow \mathbb{Q}^{\pi_0(X_Z)} \rightarrow H_{\mathcal{M}}^{2,1}(X_S) \rightarrow H_{\mathcal{M}}^{2,1}(X_U) \rightarrow 0 \end{array}$$

We then pass to the limit over all non-empty sets and use continuity for \mathbf{DA} . We obtain that $\mathrm{HP}^{n-2}(S) \rightarrow \mathrm{HP}^{n-2}(\kappa(S))$ is an isomorphism for $\neq 1, 2$, and we have a commutative diagram

$$\begin{array}{ccccccccc} 0 \rightarrow \mathrm{HP}^{-1}(X/S) \rightarrow \mathrm{HP}^{-1}(X_{\kappa(S)}/\kappa(S)) \rightarrow \mathbb{Q}^{\pi_0(X_Z)} \rightarrow \mathrm{HP}^0(X/S) \rightarrow \mathrm{HP}^0(X_{\kappa(S)}/\kappa(S)) \rightarrow 0 \\ \downarrow \quad \quad \downarrow \quad \quad \parallel \quad \quad \downarrow \quad \quad \downarrow \\ 0 \rightarrow H_{\mathcal{M}}^{1,1}(X_S) \rightarrow H_{\mathcal{M}}^{1,1}(X_{\kappa(S)}) \rightarrow \mathbb{Q}^{\pi_0(X_Z)} \rightarrow H_{\mathcal{M}}^{2,1}(X_S) \rightarrow H_{\mathcal{M}}^{2,1}(X_{\kappa(S)}) \rightarrow 0. \end{array}$$

Applying the already established result in the field case (for the function field $\kappa(S)$) and the five lemma completes the proof. \square

Remark 3.15. In the special case of $S = \mathbf{Spec}(k)$ with k a perfect field, this computation is closely related to computations of LAlb and RPic from [BVK, §9]. Let us sketch this connection. Let $f : X \rightarrow \mathbf{Spec}(k)$ be a smooth projective variety. Then X is automatically Pic-smooth, and we have proven in one key step of the proof above that the morphism

$$\omega^1 f_* \mathbb{Q}_X \rightarrow (\Sigma^\infty P(X/k))(-1)[-2].$$

induced by Θ_f is an isomorphism. By Proposition , we have the following isomorphisms.

$$\begin{aligned}\omega^1 f_* \mathbb{Q}_X &\simeq \mathbb{D}_k o^{\mathrm{tr}} \Sigma_{\mathrm{tr}}^\infty \mathrm{LAlb} \Omega_{\mathrm{tr}}^\infty a_{\mathrm{tr}} \mathbb{D}_k f_* \mathbb{Q}_X \\ &\simeq o^{\mathrm{tr}} \mathbb{D}_k^{\mathrm{tr}} \Sigma_{\mathrm{tr}}^\infty \mathrm{LAlb} M_k^{\mathrm{eff}, \mathrm{tr}}(X)\end{aligned}$$

where we have used the by now familiar comparison results to pass from \mathbf{DA} to $\mathbf{DM}^{\mathrm{eff}}$. Moreover, by Lemma 3.16 below, we can write

$$\begin{aligned}o^{\mathrm{tr}} \mathbb{D}_k^{\mathrm{tr}} \Sigma_{\mathrm{tr}}^\infty \mathrm{LAlb} M_k^{\mathrm{eff}, \mathrm{tr}}(X) &\simeq o^{\mathrm{tr}} (\Sigma^\infty \underline{\mathrm{Hom}}^{\mathrm{eff}}(\mathrm{LAlb} M_k^{\mathrm{eff}, \mathrm{tr}}(X), \mathbb{Q}(1)))(-1) \\ &\simeq o^{\mathrm{tr}} (\Sigma_{\mathrm{tr}}^\infty \mathrm{RPic}(X))(-1)\end{aligned}$$

where $\mathrm{RPic}(X)$ is the motive introduced in [BVK, Definition 8.3.1] and we have used the duality between $\mathrm{LAlb}(X)$ and $\mathrm{RPic}(X)$ coming from [BVK, §4.5]. At this point, we have an isomorphism

$$o^{\mathrm{tr}} (\Sigma_{\mathrm{tr}}^\infty \mathrm{RPic}(X))(-1) \simeq (\Sigma^\infty \mathrm{P}(X/k))(-1)[-2]$$

We now apply a^{tr} , use the isomorphism $a^{\mathrm{tr}} \Sigma^\infty \simeq \Sigma_{\mathrm{tr}}^\infty a^{\mathrm{tr}}$, Corollary 2.34, and the cancellation theorem: this yields an isomorphism

$$\mathrm{RPic}(X) \simeq \mathrm{P}^{\mathrm{tr}}(X/k)[-2].$$

We are now in position to connect with the results of [BVK]: modulo this isomorphism, the distinguished triangles of Corollary 2.34 for $\mathrm{P}^{\mathrm{tr}}(X/k)$ give an alternative proof of the computation of $\mathrm{RPic}(X)$ in [BVK, Corollary 9.6.1] in the special case X smooth projective and \mathbb{Q} -coefficients.

Lemma 3.16. *Let k be a perfect field. We have for $M \in \mathbf{DM}_{n,c}^{\mathrm{eff}}(k)$ a natural isomorphism*

$$\mathbb{D}_k^{\mathrm{tr}} \Sigma_{\mathrm{tr}}^\infty M \simeq (\Sigma^\infty \underline{\mathrm{Hom}}^{\mathrm{eff}}(M, \mathbb{Q}(n)))(-n).$$

Proof. Let $N, M \in \mathbf{DM}_c^{\mathrm{eff}}(k)$. By adjunction, monoidality of $\Sigma_{\mathrm{tr}}^\infty$ and the cancellation theorem, there is a sequence of natural isomorphisms

$$\begin{aligned}\mathbf{DM}^{\mathrm{eff}}(k)(N, \Omega_{\mathrm{tr}}^\infty \mathbb{D}_k^{\mathrm{tr}}(M(-n))) &\simeq \mathbf{DM}(k)(\Sigma_{\mathrm{tr}}^\infty N, \mathbb{D}_k^{\mathrm{tr}}(M(-n))) \\ &\simeq \mathbf{DM}(k)(\Sigma^\infty N \otimes \Sigma^\infty(M(-n)), \mathbb{Q}) \\ &\simeq \mathbf{DM}(k)(\Sigma^\infty(N \otimes M), \mathbb{Q}(n)) \\ &\simeq \mathbf{DM}^{\mathrm{eff}}(k)(N \otimes M, \mathbb{Q}(n)) \\ &\simeq \mathbf{DM}^{\mathrm{eff}}(k)(N, \underline{\mathrm{Hom}}^{\mathrm{eff}}(M, \mathbb{Q}(n)))\end{aligned}$$

which provides by the Yoneda lemma a natural isomorphism $\Omega_{\mathrm{tr}}^\infty \mathbb{D}_k^{\mathrm{tr}}(M(-n)) \simeq \underline{\mathrm{Hom}}^{\mathrm{eff}}(M, \mathbb{Q}(n))$. We apply $\Sigma_{\mathrm{tr}}^\infty$ to get an isomorphism $\Sigma_{\mathrm{tr}}^\infty \Omega_{\mathrm{tr}}^\infty \mathbb{D}_k^{\mathrm{tr}}(M(-n)) \simeq \Sigma^\infty \underline{\mathrm{Hom}}^{\mathrm{eff}}(M, \mathbb{Q}(n))$.

Moreover, the motive $\mathbb{D}_k^{\mathrm{tr}}(M(-n))$ lies in $\mathbf{DM}_c^{\mathrm{hom}}(k)$ by Proposition 1.27, Lemma 3.11 and Proposition 1.25. Because of the cancellation theorem, the counit $\Sigma_{\mathrm{tr}}^\infty \Omega_{\mathrm{tr}}^\infty \rightarrow \mathrm{id}$ is an isomorphism on $\mathbf{DM}^{\mathrm{hom}}(k)$, hence $\Sigma_{\mathrm{tr}}^\infty \Omega_{\mathrm{tr}}^\infty \mathbb{D}_k^{\mathrm{tr}}(M(-n)) \simeq \mathbb{D}_k^{\mathrm{tr}}(M(-n)) \simeq (\mathbb{D}_k^{\mathrm{tr}} M)(n)$. Combining this with the previous paragraph completes the proof. \square

Remark 3.17. The morphism Θ_f is defined for a general morphism $f : X \rightarrow S$, and, at least for some proper morphisms f , one can still prove that $\Sigma^\infty \mathrm{P}(X/S)$ lies in $\mathbf{DA}_1(S)$. However, already in the case of X singular variety over a field, the results from [BVK, §10] together with the comparison from the previous remark show that it does not correspond to $\omega^1 f_* \mathbb{Q}_X$ in general, because $\mathrm{RPic}(X)$ is not necessarily concentrated in two degrees for the t -structure of loc.cit.

In the special case of a relative curve, we can remove the regularity hypothesis on the base. This yields a general computation of the motive of a smooth projective curve.

Corollary 3.18. *Let $f : C \rightarrow S$ be a smooth projective curve. The morphism*

$$\Theta_f : (\Sigma^\infty \mathrm{P}(C/S))(-1)[-2] \rightarrow f_* \mathbb{Q}_C$$

is an isomorphism, and induces an isomorphism

$$\Sigma^\infty \mathrm{P}(C/S) \simeq M_S(C)$$

If f has a section $s : S \rightarrow C$, we have, moreover, an isomorphism

$$f_* \mathbb{Q}_C \simeq \mathbb{Q}_S \oplus \Sigma^\infty \mathrm{Jac}(C/S) \oplus \mathbb{Q}_S(1)[2].$$

Proof. Let us prove that Θ_f is an isomorphism. By [Ayo14a, Proposition 3.24], it is enough to show that $s^*\Theta_f$ is an isomorphism for any $s \in S$. By Proposition 2.38 and Proposition 2.35, we are then reduced to the case when S is the spectrum of a field. This is then a special case of Theorem 3.14.

The last statement then follows by using the section to find splittings in the distinguished triangles describing $P(C/S)$ in Corollary 2.33. \square

As an application of the computation, we can now prove a fundamental finiteness result for ω^1 .

Theorem 3.19. *Let S be a noetherian finite-dimensional excellent scheme. Assume that S admits the resolution of singularities by alterations. Then the functor $\omega^1 : \mathbf{DA}^{\text{coh}}(S) \rightarrow \mathbf{DA}^1(S)$ preserves compact objects.*

Proof. We follow the argument of [AZ12, Proposition 2.14 (vii)] for the case of ω^0 , with minor changes.

By Corollary 1.18 (ii) we can assume that S is reduced. We prove the result by noetherian induction on S . Let M be in $\mathbf{DA}_c^{\text{coh}}(S)$. Since M is compact and cohomological, Lemma 1.8 implies that there exists a finite family $\{f_i\}_{i=1}^n$ of proper morphisms $f_i : X_i \rightarrow S$ such that M lies in the triangulated subcategory generated by the motives $f_{i*}\mathbb{Q}_{X_i}$. By Proposition 2.26, there exists an everywhere dense open subset $U \subset S$ such that $f_i \times_S U$ is Pic-smooth for every i . We can moreover assume that U is regular. Write $j : U \rightarrow S$ for the open immersion and $i : Z \rightarrow S$ for the complementary reduced closed immersion. By Proposition 1.11, because of the hypothesis on resolution of singularities by alterations for S , the colocalisation triangle

$$i_*i^!M \rightarrow M \rightarrow j_*j^*M \xrightarrow{+}$$

lies in $\mathbf{DA}^{\text{coh}}(S)$. We apply ω^1 and use Proposition 3.3 (iii) to obtain a distinguished triangle

$$i_*\omega^1(i^!M) \rightarrow M \rightarrow \omega^1(j_*j^*M) \xrightarrow{+}.$$

By induction, we know that $\omega^1(i^!M)$ is compact, so it is enough to show that $\omega^1(j_*j^*M)$ is as well. By Proposition 3.3 (iii), we have an isomorphism $\omega^1(j_*j^*M) \simeq \omega^1(j_*\omega^1j^*M)$. Put $N = j_*\omega^1(j^*M)$; we have to show that $\omega^1(N)$ is compact. The motive j^*M lies in the triangulated subcategory generated by the motives $(f_i \times_S U)_*\mathbb{Q}$ with $f_i \times_S U$ Pic-smooth and U regular, hence by Theorem 3.14 we have $\omega^1(j^*M)$ compact. This implies that N is compact, with $j^*N \in \mathbf{DA}^1(U)$. In particular, we have $j_!j^*N \in \mathbf{DA}_c^1(S)$. Thus applying ω^1 to the localisation triangle for N and using Proposition 3.3 (iii) yield a distinguished triangle

$$j_!j^*N \rightarrow \omega^1N \rightarrow i_*\omega^1i^*N \xrightarrow{+}.$$

By Proposition 3.3 (vi), we have $i^*\omega^1(N) \simeq \omega^1(i^*N)$, which is compact by induction. This completes the proof that ω^1N is compact, and the proof of the theorem. \square

4. MOTIVIC t -STRUCTURES

We introduce the motivic t -structures on $\mathbf{DA}_1(S)$ and $\mathbf{DA}^1(S)$ and study how Deligne 1-motives fit in its heart.

4.1. Generated t -structures. We fix a (noetherian, finite dimensional) base scheme S for the rest of this section. We want to define t -structures by generators and relations, using the following result of Morel. We use the notations on generated subcategories of triangulated categories from the conventions section.

Proposition 4.1. [Ayo07a, Lemme 2.1.69, Proposition 2.1.70] *Let \mathcal{T} be a compactly generated triangulated category and \mathcal{G} be a family of compact objects in \mathcal{T} . Define $\mathcal{T}_{\geq 0} = \ll \mathcal{G} \gg_+$ and $\mathcal{T}_{< 0}$ as the full subcategory of all objects N with*

$$\forall n \in \mathbb{N}, \forall G \in \mathcal{G}, \text{Hom}(G, N[-n]) = 0.$$

Then $(\mathcal{T}, \mathcal{T}_{\geq 0}, \mathcal{T}_{< 0})$ is a t -structure on \mathcal{T} , that we denote by $t(\mathcal{G})$ and call the t -structure generated by \mathcal{G} .

Before we come to the generators, we need a small discussion on connected components in the relative setting, based on [Rom11]. For $f : X \rightarrow S$ morphism of schemes, let $\pi_0(X/S)$ be the functor $\mathbf{Sch}/S \rightarrow \mathbf{Ens}$ which associates to T/S the set of all open subsets U of X_T which are faithfully flat and of finite type over S and such that for all $\bar{t} \in T$ geometric point, $U_{\bar{t}}$ is a connected component of $X_{\bar{t}}$ (this is compatible with the notation introduced in Section 2.3 in the smooth projective case). By [Rom11, Theoreme 2.5.2], the functor $\pi_0(X/S)$ is representable by an étale algebraic space of finite type over S . In particular $\pi_0(X/S)$ is a constructible sheaf of sets (in the sense that it comes from a constructible sheaf of sets on the small étale site). Using the fact that such a sheaf of sets admits a stratification of S over which it is locally constant [SGA73, Exposé IX Proposition 2.5] and localisation, we see that the motive $M_S(\pi_0(X/S)) \in \mathbf{DA}(S)$ (i.e., the motive attached to the sheaf $\mathbb{Q}(\pi_0(X/S))$) is in $\mathbf{DA}_{0,c}(S)$.

Remark 4.2. Since $\pi_0(X/S)$ is an algebraic space in general, we cannot write

$$\mathbf{DA}(S)(M_S(\pi_0(X/S), M)) \simeq \mathbf{DA}(\pi_0(X/S))(\mathbb{Q}, \pi_0(X/S)^* M)$$

because the theory of $\mathbf{DA}(-)$ for algebraic spaces is not yet covered by the litterature. However, by using étale descent for $\mathbf{DA}(-)$, we see the morphism groups $\mathbf{DA}(S)(M_S(\pi_0(X/S), M))$ do behave like motivic cohomology groups, and in particular the results of Appendix B on motivic cohomology of schemes apply to these groups. We will make use of this fact at several points in this section.

Let $X \in \mathbf{Sm}/S$. There is a natural morphism $X \rightarrow \pi_0(X/S)$ which is surjective and smooth, and an induced morphism of motives $M_S(X) \rightarrow M_S(\pi_0(X/S))$. For any smooth S -scheme X , we choose a distinguished triangle

$$M_S^\perp(X) \rightarrow M_S(X) \rightarrow M_S(\pi_0(X/S)) \xrightarrow{+}.$$

The construction of this triangle commutes (up to a non-canonical isomorphism) with arbitrary base change, and we will use this fact without comment below.

We can now introduce our candidate generating families. The definition uses Deligne 1-motives over a base: for definitions and notations, we refer to the first section of Appendix A.

Definition 4.3. We define classes of objects in $\mathbf{DA}(S)$ as follows. We put

$$\mathcal{CG}_S = \{M_S(C), M_S^\perp(C)[-1] \mid C/S \text{ smooth curve}\},$$

$$\mathcal{JG}_S = \left\{ e_{\#} \Sigma^\infty(K \otimes \mathbb{Q})|e : U \rightarrow S \text{ étale}, K = \begin{matrix} \mathbb{Z} \\ \mathbb{G}_m[-1] \end{matrix} \right. \\ \left. \text{Jac}(C/U)[-1], C/U \text{ smooth projective curve} \right\},$$

and

$$\mathcal{DG}_S = \{e_{\#} \Sigma^\infty(\mathbb{M}) \mid e : U \rightarrow S \text{ étale}, \mathbb{M} \in \mathcal{M}_1(U)\}.$$

We call objects in \mathcal{CG}_S (resp. \mathcal{JG}_S , \mathcal{DG}_S) *curve generators* (resp. *Jacobian generators*, *Deligne generators*).

We are mostly interested in \mathcal{CG}_S and \mathcal{DG}_S , the family \mathcal{JG}_S is introduced as a technical intermediate.

Lemma 4.4. *The families above have the following properties.*

- (i) *Let $f : T \rightarrow S$ be a morphism of schemes. Then we have $f^* \mathcal{CG}_S \subset \mathcal{CG}_T$, $f^* \mathcal{JG}_S \subset \mathcal{JG}_T$ and $f^* \mathcal{DG}_S \subset \mathcal{DG}_T$.*
- (ii) *Let $e : T \rightarrow S$ be an étale morphism. Then $e_{\#} \mathcal{CG}_T \subset \mathcal{CG}_S$, $e_{\#} \mathcal{JG}_T \subset \mathcal{JG}_S$ and $e_{\#} \mathcal{DG}_T \subset \mathcal{DG}_S$.*

Proof. Point (i) follows from the $\mathrm{Ex}_{\#}^*$ isomorphism and Corollary 2.2. Point (ii) follows directly from the definition. \square

Lemma 4.5. *Let S be a noetherian finite dimensional scheme. We have $\mathcal{JG}_S \subset \mathcal{DG}_S$ and $\langle \mathcal{JG}_S \rangle_{(+)} \subset \langle \mathcal{CG}_S \rangle_{(+)}$.*

Proof. The first statement follows immediately from the definition. We turn to the second one. We only need to treat the $+$ variant.

Let $e : U \rightarrow S$ be an étale morphism. The motive $e_{\#} \mathbb{Q}$ is clearly in \mathcal{CG}_U . Consider the smooth curve $f : \mathbb{G}_m \times U \rightarrow U$; by Proposition 2.6, we have $M_U^\perp(\mathbb{G}_m \times U)[-1] \simeq \Sigma^\infty \mathbb{G}_m \otimes \mathbb{Q}[-1]$, which

shows that $\Sigma^\infty \mathbb{G}_m \otimes \mathbb{Q}[-1]$ is in \mathcal{CG}_U . Let $e : U \rightarrow S$ be an étale morphism and be $f : C \rightarrow U$ a smooth projective curve. By Corollary 4.1, we have an isomorphism $M_U(C) \simeq \Sigma^\infty P(C/U)$. By Corollary 2.33, the Picard complex of the curve C fits into distinguished triangles

$$R_{\pi_0(f)} \mathbb{G}_m \otimes \mathbb{Q} \rightarrow P(C/U)_{\mathbb{Q}} \rightarrow \mathcal{P}ic_{C/U}^{\text{sm}} \otimes \mathbb{Q} \xrightarrow{+}$$

and

$$\text{Jac}(C/U) \otimes \mathbb{Q} \rightarrow \mathcal{P}ic_{C/U}^{\text{sm}} \otimes \mathbb{Q} \rightarrow \pi_0(f)_\# \mathbb{Q} \xrightarrow{+}.$$

Moreover, the map $M_U(C) \rightarrow M_U(\pi_0(C/U))$ coincides modulo the isomorphism above with the composite map $\Sigma^\infty P(C/U)_{\mathbb{Q}} \rightarrow \Sigma^\infty \mathbb{Q}[\pi_0(C/U)]$. This gives us a distinguished triangle

$$M_U^\perp(C)[-1] \rightarrow \Sigma^\infty \text{Jac}(C/U) \otimes \mathbb{Q}[-1] \rightarrow \Sigma^\infty R_{\pi_0(f)} \mathbb{G}_m \otimes \mathbb{Q} \xrightarrow{+}.$$

The motive $\pi_0(f)_\# \mathbb{Q}$ is in \mathcal{CG}_U by definition. Using the compatibility between Weil restriction and pushforward (Lemma), we have $\Sigma^\infty R_{\pi_0(f)} \mathbb{G}_m \otimes \mathbb{Q} \simeq \pi_0(f)_* \Sigma^\infty (\mathbb{G}_m \otimes \mathbb{Q})$ which is in $\langle \mathcal{CG}_U \rangle_+$ as we have shown earlier in the proof. All together, this shows $\text{Jac}(C/U) \otimes \mathbb{Q}[-1]$ is in $\langle \mathcal{CG}_U \rangle_+$. Finally, in all three cases, we apply $e_\#$ and use the previous lemma to see that the result lies in $\langle \mathcal{CG}_S \rangle_+$. This shows that $\mathcal{JG}_S \subset \langle \mathcal{CG}_S \rangle_+$, as required. \square

We now come to a more difficult stability property.

Proposition 4.6. *Let $i : Z \rightarrow S$ be a closed immersion. Then*

$$i_* \langle \mathcal{JG}_Z \rangle_{(+)} \subset \langle \mathcal{JG}_S \rangle_{(+)}.$$

Proof. Let $r : Z_{\text{red}} \rightarrow Z$ be the canonical closed immersion. Localisation implies that $\text{id} \simeq r_* r^*$. Since r^* preserves \mathcal{JG} by Lemma 4.4, we see that it is enough to show the property for $i \circ r$. We can thus assume Z reduced.

We proceed by induction on the dimension of Z . If $\dim(Z) = 0$, because Z is reduced, it is a disjoint union of closed points of S . Then i_* is canonically the direct sum of the corresponding push-forwards for each point, so we can assume that Z is a single closed point $s \in S$.

There are three different types of generators in \mathcal{JG}_s . We fix $e : V \rightarrow s$ an étale morphism. Since s is a point, e is actually finite étale.

We first consider the case of a generator $e_\# \mathbb{Q}$. By standard spreading out results [Gro66, §8], there exists an open neighbourhood $s \in U \xrightarrow{e} S$ and a finite étale morphism $\tilde{e} : \tilde{V} \rightarrow U$ extending e , in the sense that we have a commutative diagram of schemes

$$\begin{array}{ccccc} \tilde{V}^\circ & \xrightarrow{\tilde{j}} & \tilde{V} & \xleftarrow{\tilde{i}} & V \\ \tilde{e}^\circ \downarrow & & \tilde{e} \downarrow & & e \downarrow \\ U \setminus s & \xrightarrow{\tilde{j}} & U & \xleftarrow{\tilde{i}} & s \end{array}$$

with cartesian squares. By localisation, we have a distinguished triangle

$$\tilde{j}_! \tilde{j}^* \tilde{e}_\# \mathbb{Q} \rightarrow \tilde{e}_\# \mathbb{Q} \rightarrow i_* \tilde{i}^* \tilde{e}_\# \mathbb{Q} \xrightarrow{+}$$

to which we apply $c_\#$ and then rewrite as

$$(c\tilde{j})_\# \tilde{e}_\#^\circ \mathbb{Q} \rightarrow c_\# \tilde{e}_\# \mathbb{Q} \rightarrow i_* e_\# \mathbb{Q} \xrightarrow{+}.$$

The motives $(c\tilde{j})_\# \tilde{e}_\#^\circ \mathbb{Q}$ and $c_\# \tilde{e}_\# \mathbb{Q}$ are in \mathcal{JG}_S , so this triangle shows that $i_* e_\# \mathbb{Q}$ lies in $\langle \mathcal{JG}_S \rangle_+$.

The case of a generator of the form $e_\# \Sigma^\infty \mathbb{G}_m \otimes \mathbb{Q} \simeq e_\# \mathbb{Q}(1)[1]$ (cf. Proposition 2.6) follows from essentially the same proof, twisting by $\mathbb{Q}(1)[1]$.

We now do the case of a generator of the form $e_\# \Sigma^\infty \text{Jac}(C/U)_{\mathbb{Q}}$ with $f : C \rightarrow U$ a smooth projective curve. We have $e_\# \Sigma^\infty \text{Jac}(C/U)_{\mathbb{Q}} \simeq e_* \Sigma^\infty \text{Jac}(C/U)_{\mathbb{Q}} \simeq \Sigma^\infty R_e \text{Jac}(C/U)_{\mathbb{Q}}$ by Lemma 4.1; the Weil restriction of the Jacobian of C along U/s is the Jacobian of C considered as a smooth projective curve over s , hence we can assume $e = \text{id}$ in what follows.

We use standard results from the deformation theory of curves. Namely, by [SGA03, Théorème 7.3, Corollaire 7.4], the curve C can be deformed to a smooth projective curve \hat{C} on $\mathbf{Spec}(\hat{\mathcal{O}}_{S,s})$. By the Artin approximation theorem, one can in fact deform C to a smooth projective curve C^h on $\mathbf{Spec}(\mathcal{O}_{S,s}^h)$ where $\mathcal{O}_{S,s}^h$ is the henselian local ring of S at s . Using spreading out results from [Gro66, §8], we arrive at the following situation. We have a pointed étale neighbourhood

$(c : U \rightarrow S, s)$ of (S, s) and a smooth projective curve $\tilde{f} : \tilde{C} \rightarrow U$ which extends C . We form the following diagram of schemes with cartesian squares

$$\begin{array}{ccccc}
 \tilde{C}^0 & \xrightarrow{\tilde{j}} & \tilde{C} & \xleftarrow{\tilde{i}} & C \\
 \downarrow \tilde{f}^\circ & & \downarrow \tilde{f} & & \downarrow \\
 U^\circ & \xrightarrow{\tilde{j}} & U & \xleftarrow{\tilde{i}} & s \\
 & \searrow c^\circ & \downarrow c & & \parallel \\
 & & S & \xleftarrow{i} & s
 \end{array}$$

We have a localisation triangle

$$\bar{j}_! \bar{j}^* \Sigma^\infty \text{Jac}(\tilde{C}/U)_\mathbb{Q} \rightarrow \Sigma^\infty \text{Jac}(\tilde{C}/U)_\mathbb{Q} \rightarrow \bar{i}_! \bar{i}^* \Sigma^\infty \text{Jac}(\tilde{C}/U)_\mathbb{Q} \xrightarrow{+}$$

to which we apply $c_\#$ and rewrite using Corollary 2.2 to obtain

$$(c^\circ)_\# \Sigma^\infty \text{Jac}(\tilde{C}^\circ/U^\circ)_\mathbb{Q} \rightarrow c_\# \Sigma^\infty \text{Jac}(\tilde{C}/U)_\mathbb{Q} \rightarrow i_* \Sigma^\infty \text{Jac}(C/s)_\mathbb{Q} \xrightarrow{+}$$

The first two terms of this complex are in \mathcal{JG}_S , so this shows that $i_* \text{Jac}(C/s)_\mathbb{Q}$ is in $\langle \mathcal{JG}_S \rangle_+$. This concludes the proof in the case $\dim(Z) = 0$.

We now come to the induction step. Let $M \in \mathcal{JG}_Z$. Write for the moment $M = e_\# \Sigma^\infty G \otimes \mathbb{Q}$ with G one of the three possible types and $e : U \rightarrow S$ étale.

Let $k : W \rightarrow Z$ be a dense open irreducible subset such that e_W is finite étale. Let $l : T \rightarrow Z$ be the complementary reduced closed immersion; let further $k' : W' \rightarrow S$ be an open immersion with $W' \cap Z = W$ and $l' : T' \rightarrow Z$ be the complementary reduced closed immersion. Write $m : W \rightarrow W'$ and $n : T \rightarrow T'$ for the induced closed immersions.

We have a localisation triangle for k, l to which we apply $i_!$ and get

$$i_! k_! k^* M \rightarrow i_* M \rightarrow i_* l_* l^* M \xrightarrow{+}$$

which can be rewritten as

$$k'_! m_! k^* M \rightarrow i_* M \rightarrow (l' \circ n)_* l^* M \xrightarrow{+}.$$

By Lemma 4.4 (i), we have $k^* M \in \mathcal{JG}_W$ and $l^* M \in \mathcal{JG}_Z$. We have $\dim(T) < \dim(Z)$ so that by induction the third term of this triangle is in $\langle \mathcal{JG}_S \rangle_+$. Moreover $k'_!$ preserves $\langle \mathcal{JG} \rangle_+$ by Lemma 4.4 (i). Together, this means that to show that $i_* M$ is in $\langle \mathcal{JG}_S \rangle_+$, we need only show that $m_! k^* M$ is in $\langle \mathcal{JG}_{W'} \rangle_+$. We are thus reduced to the case where Z is irreducible (with generic point η) and e a finite étale morphism. In that situation we can again “absorb” $e_\#$ into G and assume $e = \text{id}$ and $V = S$.

The rest of the induction step consists of applying the same type of spreading out/deformation arguments we used in the $\dim(Z) = 0$ case to G_η . Since the three cases are similar and the case of $G = \text{Jac}(C/S)$ with $f : C \rightarrow S$ smooth projective curve is the most complicated, we only detail that one.

By the same deformation argument as in the dimension 0 case, which applies to the non-closed point η as well, we can find a pointed étale neighbourhood $(e : W \rightarrow S, x \rightarrow \eta)$ of (S, η) , a smooth projective curve $\tilde{f} : \tilde{C} \rightarrow W$ which extends C_η .

Put $V = \overline{\{x\}} \subset W$ be the closure of x . By spreading-out, there exists an open neighbourhood $V^\circ \subset V$ of x and a dense open subset $Z^\circ \subset Z$ such that \tilde{f} induces an isomorphism $V^\circ \simeq Z^\circ$ (since it is an isomorphism above η). By localisation and the induction hypothesis, we can assume that $Z^\circ = Z$. We now have a smooth projective curve above an open set of S which extends f , and we can then conclude by localisation as in the end of the proof of the $\dim(Z) = 0$ case. This finishes the proof. \square

The deformation theory argument in the proof is the reason why we have introduced an arbitrary étale morphism in the definitions of \mathcal{DG} and \mathcal{JG} , instead of say an open immersion. A simpler version of the same proof yields the following 0-motivic analogue.

Lemma 4.7. *Let $i : Z \rightarrow S$ be a closed immersion. Then*

$$i_* \langle e_\# \mathbb{Q} \mid e : U \rightarrow Z \text{ étale} \rangle_{(+)} \subset \langle f_\# \mathbb{Q} \mid f : V \rightarrow S \text{ étale} \rangle_{(+)}.$$

We can now exhibit new generating families for $\mathbf{DA}_1(S)$ and $\mathbf{DA}^1(S)$.

Proposition 4.8. *Let S be a noetherian finite-dimensional scheme.*

- (i) $\langle \mathcal{CG}_S \rangle_{(+)} = \langle \mathcal{JG}_S \rangle_{(+)} = \langle \mathcal{DG}_S \rangle_{(+)}.$
- (ii)

$$\mathbf{DA}_{1,c}(S) = \langle \mathcal{CG}_S \rangle = \langle \mathcal{JG}_S \rangle = \langle \mathcal{DG}_S \rangle$$

and

$$\mathbf{DA}_1(S) = \ll \mathcal{CG}_S \gg = \ll \mathcal{JG}_S \gg = \ll \mathcal{DG}_S \gg .$$

- (iii)

$$\mathbf{DA}_c^1(S) = \langle \mathcal{CG}_S(-1) \rangle = \langle \mathcal{JG}_S(-1) \rangle = \langle \mathcal{DG}_S(-1) \rangle$$

and

$$\mathbf{DA}_1(S) = \ll \mathcal{CG}_S(-1) \gg = \ll \mathcal{JG}_S(-1) \gg = \ll \mathcal{DG}_S(-1) \gg .$$

Proof. Let us prove Point (i). Using Lemma 4.4 and localisation, we can assume that S is reduced. Lemma 4.4 already provides us with $\langle \mathcal{JG}_S \rangle_{(+)} \subset \langle \mathcal{DG}_S \rangle_{(+)}$ and $\langle \mathcal{JG}_S \rangle_{(+)} \subset \langle \mathcal{CG}_S \rangle_{(+)}$. We prove the other inclusions by noetherian induction on S . Since $\langle \mathcal{G} \rangle_{+}$ for any family \mathcal{G} , it is enough to treat the $+$ version. Let M be in either \mathcal{CG}_S or \mathcal{DG}_S . By Proposition 4.6, Lemma 4.4 and localisation, to proceed with the induction, it is enough to show that there exists a non-empty open set $j : V \rightarrow S$ such that j^*M lies in $\langle \mathcal{JG}_V \rangle_{+}$.

We first look at \mathcal{CG}_S . Let $f : C \rightarrow S$ be a smooth morphism of relative dimension ≤ 1 . Let η be a generic point of S . If η were perfect, we could use the smooth projective completion of C_η . In general, we have to be more careful. After a finite inseparable extension $h : \mathbf{Spec}(\eta') \rightarrow \mathbf{Spec}(\eta)$, the smooth curve $C_{\eta'}$ has a smooth projective completion $\bar{C}_{\eta'}$, with complement an étale η' -scheme $\partial C_{\eta'}$. By the separation property of \mathbf{DA} , we have $M_\eta(C_\eta) \simeq h_* M_{\eta'}(C_{\eta'})$. By localisation applied to the pair $(\bar{C}_{\eta'}, C_{\eta'})$, we get a distinguished triangle

$$h_* M_{\eta'}(\partial C_{\eta'})(1)[1] \rightarrow h_* M_{\eta'}(C_{\eta'}) \rightarrow h_* M_{\eta'}(\bar{C}_{\eta'}) \xrightarrow{+}.$$

By Lemma 1.26, there exists a finite étale morphism D/η (resp. a smooth projective curve \tilde{C}/η) such that $h_* M_{\eta'}(\partial C_{\eta'}) \simeq M_\eta(D)$ (resp. $h_* M_{\eta'}(\bar{C}_{\eta'}) \simeq M_\eta(\tilde{C})$). Putting this together, we get a distinguished triangle

$$M_\eta(D)(1)[1] \rightarrow M_\eta(C_\eta) \rightarrow M_\eta(\tilde{C}) \xrightarrow{+}.$$

Moreover, by spreading out, we can find a normal open subset $\eta \in V \subset S$ such that $\partial \tilde{C}$ (resp. \tilde{C}) extend to a finite étale morphism (resp. a smooth projective morphism) over V and a distinguished triangle

$$M_V(D)(1)[1] \rightarrow M_V(C_V) \rightarrow M_V(\tilde{C}) \xrightarrow{+}.$$

This triangle, together with Corollary 2.9 applied to $R_{D/V}\mathbb{G}_m$ and Corollary 4.1 applied to \tilde{C} , shows that $M_V(C_V)$ is in $\ll \mathcal{JG}_V \gg_{+}$.

We have $\pi_0(C_\eta/\eta) \simeq h_* \pi_0(C_{\eta'}/\eta')$ because h is finite inseparable. We have $\pi_0(C_{\eta'}/\eta') \simeq \pi_0(\bar{C}_{\eta'}/\eta')$ because this is true in general of the smooth projective completion of a smooth curve over a perfect field. Finally, from $h_* M_{\eta'}(\bar{C}_{\eta'}) \simeq M_\eta(\tilde{C}_\eta)$ and the computation of the motive of a smooth projective curve (Corollary), we deduce that $h_* \pi_0(\bar{C}_{\eta'}/\eta') \simeq \pi_0(\tilde{C}_\eta/\eta)$. All together, this shows that $\pi_0(C_\eta/\eta) \simeq \pi_0(\tilde{C}_\eta/\eta)$. This implies by continuity that by restricting to a smaller V we can ensure that

$$\pi_0(C_V/V) \simeq \pi_0(\tilde{C}_V/V).$$

In particular, we have a distinguished triangle

$$M_V(D)(1) \rightarrow M_V^\perp(C_V)[-1] \rightarrow M_V^\perp(\tilde{C})[-1] \xrightarrow{+}.$$

The motive $M_V(D)(1)$ is in $\ll \mathcal{JG}_V \gg_{+}$. It remains to show the same for $M_V^\perp(\tilde{C})[-1]$. By Corollary 4.1, we have $M_V(\pi_0(\tilde{C}/V)) \simeq \Sigma^\infty \mathrm{NS}_{\tilde{C}/V}^{\mathrm{sm}} \otimes \mathbb{Q}$ and we find a distinguished triangle

$$\pi_0(\tilde{C}/V)_* \mathbb{Q}(1)[1] \rightarrow M_V^\perp(\tilde{C})[-1] \rightarrow \Sigma^\infty \mathrm{Jac}(\tilde{C}/V) \otimes \mathbb{Q}[-1] \xrightarrow{+}.$$

which shows that $M_V^\perp(C_V)[-1]$ lies in $\ll \mathcal{JG}_V \gg_{+}$. We have achieved our goal.

We now look at \mathcal{DG}_S . A lattice (resp. a torus) on a reduced scheme is generically a direct factor of a permutation lattice (resp. torus) by [SGA70, Exp. X 6.2], while an abelian scheme on S is

generically and up to isogeny a direct factor of a relative Jacobian by [Kat99, Theorem 11] applied at a generic point and a spreading out argument. This implies that for any $M \in \mathcal{DG}_S$, there exists a non-empty open $j : V \rightarrow S$ such that j^*M is a direct factor of a motive in \mathcal{JG}_V . This completes the proof of Point (i).

For Point (ii), we only have to notice that by definition (resp. by Lemma 1.8) we have $\mathbf{DA}_1(S) = \ll \mathcal{CG}_S \gg$ (resp. $\mathbf{DA}_{1,c}(S) = \langle \mathcal{CG}_S \rangle$) and the rest then follows from Point (i). Finally, Point (iii) is deduced from (ii) using Proposition 1.27. \square

We come to the main definition of this paper.

Definition 4.9. The *motivic t-structure* $t_{\mathbf{MM},1}(S)$ on $\mathbf{DA}_1(S)$ (resp. $t_{\mathbf{MM}}^1(S)$ on $\mathbf{DA}^1(S)$) is the t-structure $t(\mathcal{CG}_S)$ (resp. $t(\mathcal{CG}_S(-1))$). The heart of $t_{\mathbf{MM},1}$ (resp. $t_{\mathbf{MM}}^1$) is the *abelian category of 1-motivic sheaves* $\mathbf{MM}_1(S)$ (resp. $\mathbf{MM}^1(S)$).

The two abelian categories $\mathbf{MM}_1(S)$ and $\mathbf{MM}^1(S)$ are isomorphic via Tate twists, but embedded differently in $\mathbf{DA}(S)$. From Proposition 4.8 it follows immediately that

Corollary 4.10. $t_{\mathbf{MM},1} = t(\mathcal{JG}_S) = t(\mathcal{DG}_S)$ (resp. $t_{\mathbf{MM}}^1 = t(\mathcal{JG}_S(-1)) = t(\mathcal{DG}_S(-1))$).

We also introduce a parallel definition for 0-motives.

Definition 4.11. The *motivic t-structure* $t_{\mathbf{MM},0}(S) = t_{\mathbf{MM}}^0(S)$ on $\mathbf{DA}_0(S) = \mathbf{DA}^0(S)$ is the t-structure generated by the family of objects of the form $e_{\sharp}\mathbb{Q}$ with $e : T \rightarrow S$ étale. The heart of $t_{\mathbf{MM}}^0$ is the *abelian category of 0-motivic sheaves* $\mathbf{MM}^0(S)$.

Remark 4.12. The t-structure $t_{\mathbf{MM},0}(S)$ is somewhat similar to the *homotopy t-structure* on the whole of $\mathbf{DA}(S)$, which we define, following [Ayo07a, Definition 2.2.41], as the t-structure generated by the objects $f_{\sharp}\mathbb{Q}(n)[n]$ for all $f : T \rightarrow S$ smooth and $n \in \mathbb{Z}$. We conjecture that the homotopy t-structure restricts to $\mathbf{DA}_0(S)$ and that its restriction is $t_{\mathbf{MM},0}(S)$; this can be shown in the case when $S = \mathbf{Spec}(k)$ with k a field by first comparing with \mathbf{DM} over the perfect closure and using some results from [Dég11].

We now discuss some elementary exactness properties of Grothendieck operations with respect to the motivic t-structures.

Proposition 4.13. *The following properties hold for $t_{\mathbf{MM},1}$, $t_{\mathbf{MM}}^1$ and $t_{\mathbf{MM},0}$.*

- (i) *Let f be a morphism of schemes; then f^* is t-positive.*
- (ii) *Let f be a quasi-finite separated morphism; then $f_!$ is t-positive.*
- (iii) *Let e be an étale morphism; then e^* is t-exact.*
- (iv) *Let f be a finite morphism; then f_* is t-exact.*

Let $\epsilon \in \{0, 1\}$; the following properties hold for $t_{\mathbf{MM}}^{\epsilon}$.

- (i) *Let f be a morphism of schemes; then $\omega^{\epsilon}f_*$ is t-negative.*
- (ii) *Let f be a quasi-finite separated morphism of schemes; then $\omega^{\epsilon}f^!$ is t-negative.*

Proof. By Proposition 1.17 (resp. 1.16) and the very definition of ω^0 and ω^1 , all the operations above are well-defined. We prove the proposition for $t_{\mathbf{MM},1}$; the proof for the corresponding statements for $t_{\mathbf{MM}}^1$ is then obtained by twisting by $\mathbb{Q}(-1)$, and the proof for $t_{\mathbf{MM}}^0$ is completely analogous (using Lemma instead of Proposition 4.6)

Let $f : S \rightarrow T$ be any morphism of schemes. Then f^* , $f_!$ both commute with small sums since they are left adjoints. By [Ayo07a, Lemme 2.1.78], to prove statements (i), (ii), it remains to show that $f^*\mathcal{DG}_T \subset \mathbf{DA}_1(S)_{\geq 0}$ and that when f is quasi-finite, $f_!\mathcal{DG}_S \subset \mathbf{DA}_1(S)_{\geq 0}$.

In the case of f^* , we deduce from the $\mathrm{Ex}_!^*$ isomorphism and Proposition 2.2 that we have the inclusion $f^*\mathcal{DG}_T \subset \mathcal{DG}_S$. This proves (i).

For the case of $f_!$, we proceed in several steps. If e is an étale morphism, we have $e_!\mathcal{DG}_S \subset \mathcal{DG}_T$ by definition. If i is a closed immersion, we have $i_!\mathcal{DG}_S \subset \mathcal{DG}_T$ by Proposition 4.6 and Proposition 4.8. Let f be an arbitrary quasi-finite morphism. At this point, we have that for an open immersion j (resp. a closed immersion i), the functors $j_!$ and j^* (resp. the functors $i_!$ and i^*) are t-positive. This shows that to prove that an object M is t-positive, one can proceed by localisation. A noetherian induction together with the étale case above then reduce us to the case where f is finite surjective radicial, and allows us further to restrict to an arbitrary dense open set of the base. Using continuity, this reduces us to the field case, where we can apply Lemma 1.26.

Let f be an étale morphism (resp. a finite morphism). We have seen above that f^* (resp. $f_* \simeq f_!$) is t -positive. Moreover, since $e_! \simeq e_*$ (resp. f^*) is t -positive, its right adjoint e^* (resp. f_*) is t -negative. This proves (iii) (resp. (iv)).

Let $f : S \rightarrow T$ be a morphism (resp. a quasi-finite separated morphism). We have seen above that $f^* : \mathbf{DA}^1(T) \rightarrow \mathbf{DA}^1(S)$ (resp. $f_! : \mathbf{DA}^1(S) \rightarrow \mathbf{DA}^1(T)$) is t -positive, so its right adjoint $\omega^1 f_*$ (resp. $\omega^1 f_!$) is t -negative. This proves (i) (resp. (ii)). \square

From the definition, we also get a partial result about the Betti and ℓ -adic realisation functors. Recall from [Ayo10] that for $\sigma : k \rightarrow \mathbb{C}$ a field of characteristic 0 with a fixed complex embedding $\sigma : k \rightarrow \mathbb{C}$ and S scheme of finite type over k , there is a (covariant) functor

$$R_{B,\sigma} : \mathbf{DA}(S) \rightarrow D(S^{\text{an}}, \mathbb{Q})$$

with target the derived category of sheaves of \mathbb{Q} -vector spaces on the complex analytic space S^{an} . Note that in [Ayo10] this functor is only defined in the case S quasi-projective over k , but this is due solely to quasi-projectivity hypotheses in the theory of the six operations for $\mathbf{DA}(S)$ in [Ayo07a] which can be removed using results of [CD].

Similarly, we fix a prime ℓ , and let S be a $\mathbb{Z}[\frac{1}{\ell}]$ -scheme. Let $D_c(S, \mathbb{Q}_\ell)$ be subcategory of complexes with constructible cohomology in the derived category of \mathbb{Q}_ℓ -sheaves S in the sense of Ekedahl [Eke90]. By [Ayo14a, Section 9], there is a covariant functor

$$R_\ell : \mathbf{DA}_c(S) \rightarrow D_c(S, \mathbb{Q}_\ell).$$

Unfortunately, the “unbounded” ℓ -adic realisation with source $\mathbf{DA}(S)$, presumably with natural target category defined using the pro-étale topology of [BS13], has not been constructed yet.

Proposition 4.14. *• Let k be a field with a fixed complex embedding σ and S be a scheme of finite type over k . The functor $R_{B,\sigma}$, restricted to either $\mathbf{DA}_0(S)$, $\mathbf{DA}_1(S)$ or $\mathbf{DA}^1(S)$ is t -positive with respect to the motivic t -structure and the standard t -structure.*
• Let ℓ be a prime, and let S be a $\mathbb{Z}[\frac{1}{\ell}]$ -scheme. The functor R_ℓ , restricted to either $\mathbf{DA}_{0,c}(S)$, $\mathbf{DA}_{1,c}(S)$ or $\mathbf{DA}_c^1(S)$, sends compact $t_{\mathbf{MM}}$ -positive objects to positive objects in the standard t -structure.

Proof. Because of the definition of the motivic t -structures above, and the structure of t -positive and compact objects in a generated t -structure, it is enough to show that the image of the compact generators is t -positive for the standard t -structure. The three cases being similar, let us treat the one of $\mathbf{DA}_1(S)$. By Corollary 4.10, we can use the family \mathcal{DG}_S . Let $e : U \rightarrow S$ be an étale morphism, $\mathbb{M} = [L \rightarrow G] \in \mathcal{M}_1(U)$ and $M = e_! \Sigma_U^\infty \mathbb{M} \in \mathcal{DG}_U$ (recall that $e_! \simeq e_*$ because e is étale).

Write R for either R_B or R_ℓ (with the appropriate hypothesis on S). Then $RM \simeq e_! R(\Sigma_U^\infty \mathbb{M})$ with $e_!$ the corresponding Grothendieck operations on derived categories of sheaves (by [Ayo10, Theoreme 3.19] for $R = R_B$ and [Ayo14a, Theoreme 9.7] for $R = R_\ell$). Since the functor $e_!$ is then t -exact for the standard t -structures, we only need to show that $R(\Sigma_U^\infty \mathbb{M})$ is t -positive. Let us show that it is in fact in the heart of the standard t -structure. We can show this separately for $\mathbb{M} = [L \rightarrow 0]$ and $\mathbb{M} = [0 \rightarrow G]$, i.e., we need to compute $R(\Sigma^\infty L)$ and $R(\Sigma^\infty G[-1])$.

Note that because of the commutation of R with the six operations, localisation and the t -exactness of $j_! j^*$ and $i_* i^*$ for the standard t -structures, we can always restrict to a non-empty open set of U and argue by noetherian induction. We can then assume U_{red} to be normal, and then write L as a direct factor of $h_! \mathbb{Q}_T$ for $h : T \rightarrow U$ finite étale. Applying again the commutation of R with $h_!$ and the t -exactness of $h_!$ for the standard t -structures, we conclude that $R(\Sigma^\infty L)$ is in the heart.

In the case of $\Sigma^\infty G[-1]$, our claim follows from the computation of the realisation of such a motive in [AHPL14, Proposition 5.1.(2)] (for R_B) and [AHPL14, 5.2] (for R_ℓ). This completes the proof. \square

There are simple connections between the t -structures for 0 and 1-motives.

Proposition 4.15. *Let S be a noetherian finite-dimensional scheme.*

- (i) *The inclusions of $\mathbf{DA}_0(S)$ into $\mathbf{DA}_1(S)$.*
- (ii) *The t -structure $t_{\mathbf{MM},1}(S)$ restricts to $\mathbf{DA}_0(S)$, and its restriction coincide with $t_{\mathbf{MM},0}(S)$.*

Proof. Let us prove Statement (i). The inclusion functor commute with small sums. The generators $e_{\sharp}\mathbb{Q}$ ($e : U \rightarrow S$ étale) of $t_{\mathbf{MM}}^0$ are also t -positive for $t_{\mathbf{MM},1}$; this implies that the inclusion is t -positive.

Let us now show the inclusion $\mathbf{DA}_1(S)$ is t -negative. Let $N \in \mathbf{DA}^0(S)_{\leq 0}$. Using Proposition 4.8 (i), we see that we have to show that for every étale morphism $e : U \rightarrow S$, $\mathbb{M} = [L \rightarrow G] \in \mathcal{M}_1(U)$, and $n \in \mathbb{N}^*$, we have $\mathbf{DA}(S)(e_{\sharp}(\Sigma^{\infty}\mathbb{M})[n], N) = 0$. Using the $e_{\sharp} \dashv e^*$ adjunction and the fact that e^* is t -negative (Proposition 4.13 (iii)), we reduce to the case $e = \text{id}$. We have a distinguished triangle

$$\Sigma^{\infty}G_{\mathbb{Q}}[-1] \rightarrow \Sigma^{\infty}\mathbb{M} \rightarrow \Sigma^{\infty}L_{\mathbb{Q}} \xrightarrow{+}.$$

Let us show that, for all $P \in \mathbf{DA}^0(S)$, we have $\mathbf{DA}(S)(\Sigma^{\infty}G_{\mathbb{Q}}, P) = 0$. Because $\Sigma^{\infty}G$ is compact, this vanishing statement can be checked on compact generators, so we can assume that P is of the form $a_*\mathbb{Q}_X[m]$ for some $a : X \rightarrow S$ finite and $m \in \mathbb{Z}$. Using the $a^* \dashv a_*$ adjunction and Proposition 2.2, we see that we can assume $a = \text{id}$, so we have to show that $\mathbf{DA}(S)(\Sigma^{\infty}G_{\mathbb{Q}}, \mathbb{Q}[m]) = 0$. By [AHPL14, Theorem 3.3], $\Sigma^{\infty}G_{\mathbb{Q}}$ is a direct factor of $M_S(G)$, characterised as the n -eigenspace for the morphism induced by $[n]_G$ for any $n \neq 1$, and that $M_S(G)$ has also a direct factor \mathbb{Q}_S , characterised as the 1-th eigenspace for $[n]_G$. We have $\mathbf{DA}(S)(M_S(G), \mathbb{Q}[m]) \simeq H_{\mathcal{M}}^{m,0}(G)$; since $\pi : G \rightarrow S$ is smooth surjective with connected fibers, we deduce by Proposition B.5 (iv) that $\pi^* : H_{\mathcal{M}}^{m,0}(S) \rightarrow H_{\mathcal{M}}^{m,0}(G)$ is an isomorphism. Looking at the action of $[n]_G$, this shows that all the weight 0 motivic cohomology of G comes from the direct factor \mathbb{Q}_S of $M_S(G)$, and accordingly we deduce that $\mathbf{DA}(S)(\Sigma^{\infty}G_{\mathbb{Q}}, \mathbb{Q}[m]) = 0$ as claimed. This shows that $\mathbf{DA}(S)(\Sigma^{\infty}\mathbb{M}[n], N) \simeq \mathbf{DA}(S)(\Sigma^{\infty}L_{\mathbb{Q}}[n], N)$.

On the other hand, the motive $\Sigma^{\infty}L(-1)$ is in $\mathbf{DA}_0(S)$ and $t_{\mathbf{MM},0}$ -positive; this would be clear for S normal since L is then a direct factor of a permutation lattice, in general this can be checked by noetherian induction starting from a non-empty open set $V \subset U$ with V_{red} normal, using localisation, Proposition 2.2 and Proposition 4.13. Since by hypothesis N is $t_{\mathbf{MM},0}$ -negative, we have $\mathbf{DA}(S)(\Sigma^{\infty}L_{\mathbb{Q}}[n], N) = 0$. This completes the proof that $\mathbf{DA}_0(S) \rightarrow \mathbf{DA}_1(S)$ is t -negative, hence t -exact.

We now prove Statement (ii). Let us start with $t_{\mathbf{MM},1}$. Write ${}_{0}\tau_{\geq 0}$ and ${}_{1}\tau_{\geq 0}$ for the truncation functors of $t_{\mathbf{MM},0}$ and $t_{\mathbf{MM},1}$. We have to show that for every $M \in \mathbf{DA}_0(S)$, we have ${}_{1}\tau_{\geq 0}M \in \mathbf{DA}_0(S)$ and ${}_{1}\tau_{\geq 0}M \simeq {}_{0}\tau_{\geq 0}M$. But this follows immediately from the t -exactness of the inclusion, proved above. \square

Remark 4.16. We also conjecture the same result holds for $t_{\mathbf{MM}}^1(S)$, but this seems to require more delicate vanishing results. We leave this problem to the interested reader.

4.2. The t -structures over a field. In this short section, we compare our t -structures for homological 0 and 1-motives with the existing work on t -structures for $\mathbf{DM}_0^{\text{eff}}(k)$ and $\mathbf{DM}_1^{\text{eff}}(k)$ with k a perfect field [Org04] [Ayo11], and we extend the results from these references to a possibly imperfect field.

For clarity, let us treat first the simple case of 0-motives. Let k be a perfect field. First, let us reformulate the treatment in [Org04, §2]. There is a functor $\mathbf{Sh}_{\text{ét}}(k, \mathbb{Q}) \rightarrow \mathbf{DM}^{\text{eff}}(k)$ (any sheaf of \mathbb{Q} -vector spaces on the small étale site has a canonical extension as an étale sheaf with transfers on \mathbf{Sm}/k) which extends to a triangulated functor $D(\mathbf{Sh}_{\text{ét}}(k, \mathbb{Q})) \rightarrow \mathbf{DM}^{\text{eff}}(k, \mathbb{Q})$. This factors through $\mathbf{DM}_0^{\text{eff}}(k)$, and the resulting functor is an equivalence of categories $\mathcal{R}_{\text{tr}}^{\text{eff},0} : D(\mathbf{Sh}_{\text{ét}}(k)) \simeq \mathbf{DM}_0^{\text{eff}}(k)$.

Another approach consists in first introducing the *homotopy t -structure* on $\mathbf{DM}^{\text{eff}}(k)$; this is the t -structure induced on $\mathbf{DM}^{\text{eff}}(k)$ from the standard t -structure on $D(\mathbf{Sh}((\text{Cor}/k)_{\text{ét}}, \mathbb{Q}))$, but for our purposes it is best described as the t -structure on the compactly generated triangulated category $\mathbf{DM}^{\text{eff}}(k)$ by the family of objects of the form $M_k^{\text{eff},\text{tr}}(X)$ for all $X \in \mathbf{Sm}/k$ [Ayo11, Proposition 3.3]. We claim that the homotopy t -structure restricts to $\mathbf{DM}_0^{\text{eff}}(k)$, and that the restriction coincides with the t -structure generated by the family of objects of the form $M_k^{\text{eff},\text{tr}}(Y)$ for all Y/k finite étale. To do this, it suffices that the inclusion functor $\mathbf{DM}_0^{\text{eff}}(k) \rightarrow \mathbf{DM}^{\text{eff}}(k)$ is t -exact for those two t -structures; it is t -positive because of the inclusion of generators, and t -negative because its left adjoint $L\pi_0$ is t -positive since $L\pi_0((M_k^{\text{eff},\text{tr}}(X))) \simeq M_k^{\text{eff},\text{tr}}(\pi_0(X/k))$ for any X/k smooth.

It is easy to see that the t -structures on $\mathbf{DM}_0^{\text{eff}}(k)$ introduced in the two previous paragraphs coincide. Moreover, through the equivalence of categories of Lemma 3.11, we get an equivalence of

categories $\mathcal{R}^0 : D(\mathbf{Sh}_{\text{ét}}(k, \mathbb{Q})) \rightarrow \mathbf{DA}_0(k)$, and this is an equivalence of t -categories when we equip $\mathbf{DA}_0(k)$ with $t_{\mathbf{MM},0}$.

Finally, these t -structures on $\mathbf{DM}_0^{\text{eff}}(k)$ and $\mathbf{DA}_0(k)$ restrict to compact objects; more precisely, there are equivalences of categories $D^b(\mathbf{Sh}_c(k, \mathbb{Q})) \simeq \mathbf{DM}_{0,c}^{\text{eff}}(k) \simeq \mathbf{DA}_{0,c}(k)$ and the restriction of the t -structure coincides with the standard t -structure on the bounded derived category.

Let now k be a general field and let $h : \mathbf{Spec}(k^{\text{perf}}) \rightarrow \mathbf{Spec}(k)$ be a perfect closure. We have a commutative diagram

$$\begin{array}{ccc} D(\mathbf{Sh}_{\text{ét}}(k)) & \xrightarrow{\mathcal{R}^0} & \mathbf{DA}_0(k) \\ h^* \downarrow \sim & & h^* \downarrow \sim \\ D(\mathbf{Sh}_{\text{ét}}(k^{\text{perf}}, \mathbb{Q})) & \xrightarrow[\mathcal{R}^0]{\sim} & \mathbf{DA}_0(k^{\text{perf}}) \end{array}$$

where the bottom horizontal functor is an equivalence by the case of a perfect field, the left vertical functor is an equivalence because the étale sites of k and k^{perf} are canonically isomorphic via h , and the right vertical functor is an equivalence by the separation property of $\mathbf{DA}(-)$ and Corollary 1.18 (ii). Moreover, the functor $h^* : D(\mathbf{Sh}_{\text{ét}}(k)) \rightarrow D(\mathbf{Sh}_{\text{ét}}(k^{\text{perf}}))$ is clearly t -exact, the functor $h^* : \mathbf{DA}_0(k) \rightarrow \mathbf{DA}_0(k^{\text{perf}})$ is t -exact because it is the inverse of the t -exact functor h_* (Proposition 4.13 (iv)), and $\mathcal{R}^0(k^{\text{perf}})$ is t -exact by the perfect field case. This proves that the top arrow is also an equivalence of t -categories. There is a similar diagram in the compact case which we will not spell out. Let us summarize the results so far.

Proposition 4.17. *Let k be a field. The t -structure $t_{\mathbf{MM},0}$ restricts to compact objects, and we have equivalences of t -categories*

$$\begin{aligned} \mathcal{R}^0 : (D(\mathbf{Sh}_{\text{ét}}(k, \mathbb{Q})), \text{std}) &\longrightarrow (\mathbf{DA}_0(k), t_{\mathbf{MM},0}) \\ \mathcal{R}^0 : (D^b(\mathbf{Sh}_c(k, \mathbb{Q})), \text{std}) &\longrightarrow (\mathbf{DA}_0(k), t_{\mathbf{MM},0}) \end{aligned}$$

We now turn to the case of 1-motives. Assume again momentarily that k is a perfect field. By [BVK, Lemma 1.4.4], for any commutative locally of finite type k -group scheme G , the sheaf represented by G on \mathbf{Sm}/k has a canonical structure of étale sheaf with transfers. Write G^{tr} for this sheaf with transfers, with $\varrho^{\text{tr}} G^{\text{tr}} \simeq G$.

Applying this construction at the level of complexes, Orgogozo defines in [Org04, 3.3.2] a functor which we will denote by

$$\mathcal{R}_1^{\text{eff}, \text{tr}} : \mathcal{M}_1(k) \rightarrow \mathbf{DM}_c^{\text{eff}}(k).$$

The category $\mathcal{M}_1(k)$ is in this situation an abelian category [Org04, Lemme 3.2.2] and this functor can in fact be extended to a functor

$$\mathcal{R}_1^{\text{eff}, \text{tr}} : D^b(\mathcal{M}_1(k)) \rightarrow \mathbf{DM}_c^{\text{eff}}(k).$$

This functor factors through $\mathbf{DM}_{1,c}^{\text{eff}}(k)$ (denoted as iso $d_1 \mathbf{DM}_{\text{gm}}^{\text{eff}}(\eta)$ in loc. cit.) and the resulting functor is then an equivalence of categories [Org04, Theorem 3.4.1]. In particular, this provides a t -structure on $\mathbf{DM}_{1,c}^{\text{eff}}(k)$, which we will denote by $t_1^{\text{Or}}(k)$.

In [Ayo11, Definition 3.5], Ayoub defines a t -structure on $\mathbf{DM}^{\text{eff}}(k)$ which he call the 1-motivic t -structure and which we will denote by $t_1^{\text{Ay}}(k)$. For our purposes, and according to [Ayo11, Proposition 3.7], it can be defined as the t -structure generated by objects of the form $M_k^{\text{eff}, \text{tr}}(X)$ and $M_k^{\text{eff}, \text{tr}}(X)^{\perp}[-1]$ for every $X \in \mathbf{Sm}/k$ (where $M_k^{\text{eff}, \text{tr}}(X)^{\perp}$ is a $\text{Ker}(\mathbb{Q}_{\text{tr}}(X) \rightarrow \mathbb{Q}_{\text{tr}}(\pi_0(X))) \in \mathbf{DM}^{\text{eff}}(k)$). Let us write $t_1^{\text{Ay}'}(k)$ for the t -structure on $\mathbf{DM}_1^{\text{eff}}(k)$ generated by objects of the form $M_k^{\text{eff}, \text{tr}}(C)$ and $M_k^{\text{eff}, \text{tr}}(C)^{\perp}[-1]$ for every smooth curve C/k .

Through the equivalences of Lemma 3.11, the t -structure $t_1^{\text{Ay}'}(k)$ corresponds to the t -structure $t_{\mathbf{MM},1}(k)$ since they have corresponding families of generators. This observation in fact motivated the definition of $t_{\mathbf{MM},1}(k)$ (and its generalisation to $t_{\mathbf{MM},1}(S)$). By [AHPL14,], we have moreover $a_{\text{tr}} G \simeq G^{\text{tr}}$ in $\mathbf{DM}^{\text{eff}}(k)$, which provides a natural isomorphism $\Sigma_{\text{tr}}^{\infty} \mathcal{R}^{\text{eff}, \text{tr}} \simeq a_{\text{tr}} \mathcal{R}$. Translating Proposition 4.8 and Lemma 4.1 to $\mathbf{DM}^{\text{eff}}(k)$ via the equivalences and using, we deduce that the t -structure $t_1^{\text{Ay}'}(k)$ is also generated by objects of the form $\mathcal{R}^{\text{eff}, \text{tr}} M$ for $M \in \mathcal{M}_1(k)$. It is then easy to see that the inclusion functor $(\mathbf{DM}_{1,c}^{\text{eff}}(k), t_1^{\text{Or}}(k)) \rightarrow (\mathbf{DM}_1^{\text{eff}}(k), t_1^{\text{Ay}'}(k))$ is t -exact, which implies that the t -structure $t_1^{\text{Ay}'}(k)$ restricts to compact objects, with restriction $t_1^{\text{Or}}(k)$.

As an aside, we notice the following compatibility, which could have figured in [Ayo11].

Lemma 4.18. *The t -structure $t_1^{\text{Ay}}(k)$ restricts to $\mathbf{DM}_1^{\text{eff}}(k)$, and the resulting t -structure is $t_1^{\text{Ay}'}(k)$.*

Proof. It is enough to show that the fully faithful inclusion $\mathbf{DM}_1^{\text{eff}}(k) \rightarrow \mathbf{DM}^{\text{eff}}(k)$ is t -exact. It is clearly t -positive because of the inclusion of generators. To show that it is t -negative, we may show that its left adjoint LAlb is t -positive. Let X/k be a smooth variety. By [BVK, Theorem 9.2.3], we have distinguished triangles

$$\text{NS}_{X/k}^* \otimes \mathbb{Q}[1] \rightarrow \text{LAlb}(X) \rightarrow \mathcal{A}_{X/k} \otimes \mathbb{Q} \xrightarrow{+}$$

and

$$\mathcal{A}_{X/k}^0 \otimes \mathbb{Q} \rightarrow \mathcal{A}_{X/k} \otimes \mathbb{Q} \rightarrow \mathbb{Q}[\pi_0(X/k)] \xrightarrow{+}$$

with $\text{NS}_{X/k}^*$ the “Néron-Severi torus” and $\mathcal{A}_{X/k}^0$ the semi-abelian Albanese variety of X . Because $t_1^{\text{Ay}'}(k)$ is generated by Deligne 1-motives, this shows that $\text{LAlb}(M_k^{\text{eff}, \text{tr}}(X))$ is positive. Comparing the triangles for X and $\pi_0(X)$, we also see that $\text{LAlb}(M_k^{\text{eff}, \text{tr}}(X)^\perp[-1])$ is also positive. This completes the proof. \square

Let now k be a general field and $h : \mathbf{Spec}(k^{\text{perf}}) \rightarrow \mathbf{Spec}(k)$ be a perfect closure. The functor $h^* : (\mathbf{DA}_1(k), t_{\text{MM},1}) \rightarrow (\mathbf{DA}_1(k^{\text{perf}}), t_{\text{MM},1})$ is an equivalence of t -categories by the separation property of $\mathbf{DA}(-)$, Corollary 1.18 (ii), and Proposition 4.13 (iv). It follows that $t_{\text{MM},1}(k)$ restricts to compact objects.

There is a commutative diagram

$$\begin{array}{ccc} (D^b(\mathcal{M}_1(k)), \text{std}) & \xrightarrow{\mathcal{R}} & (\mathbf{DA}_{1,c}(k), t_{\text{MM},1}) \\ h^* \downarrow \sim & & h^* \downarrow \sim \\ (D^b(\mathcal{M}_1(k^{\text{perf}})), \text{std}) & \xrightarrow{\mathcal{R}^0} & (\mathbf{DA}_{1,c}(k^{\text{perf}}), t_{\text{MM},1}) \end{array}$$

where the left vertical functor is an equivalence of t -categories (this reduces to the case of Frobenius, which then follows because we have inverted $\text{char}(k)$), the right vertical functor was just shown to be an equivalence of t -categories, and the bottom is an equivalence of categories by the perfect field case. We deduce that \mathcal{R}_k is an equivalence of t -categories for any k . Let us summarize the results for 1-motives.

Proposition 4.19. *Let k be a field. The t -structure $t_{\text{MM},1}$ restricts to compact objects, and we have an equivalence of t -categories*

$$\Sigma^\infty : (D^b(\mathcal{M}_1(k)), \text{std}) \longrightarrow (\mathbf{DA}_{1,c}(k), t_{\text{MM},1}).$$

Remark 4.20. In fact, the heart of $t_{\text{MM},1}$ is equivalent to $\text{Ind}(\mathcal{M}_1(k))$ [Ayo11, Remark 3.12] and it is plausible that the methods of that paper can be extended to show that $\mathbf{DA}_1(k) \simeq D(\text{Ind}(\mathcal{M}_1(k)))$. We leave this question to the interested reader.

For future reference, we single out the following computation, which is deduced by the comparison results above from [Org04, Proposition 3.3.3] and [Org04, Proposition 3.2.4].

Proposition 4.21. *Let k be a field, $M_1, M_2 \in \mathcal{M}_1(k)$ and $n \in \mathbb{Z}$. Then*

$$\begin{aligned} \mathbf{DA}(k)(\Sigma^\infty M_1, \Sigma^\infty M_2[n]) &\simeq \text{Ext}_{\mathcal{M}_1(k)}^n(M_1, M_2) \\ &\simeq 0, \neq 0, 1. \end{aligned}$$

4.3. Deligne 1-motives and the heart. In this section, we compute certain morphism groups between objects in $\mathbf{DA}_1(S)$ and $\mathbf{DA}^1(S)$ and deduce various properties of the motivic t -structure.

The following theorem shows the advantage of the Deligne generating family: it lies in the heart of the motivic t -structure.

Theorem 4.22. *Let S be a noetherian finite-dimensional scheme. We have $\mathcal{DG}_S \subset \mathbf{MM}_1(S)$ (resp. $\mathcal{DG}_S(-1) \subset \mathbf{MM}^1(S)$).*

Proof. We have shown in Proposition 4.8 that the generators are t -positive, it remains to show that they are t -negative. Using the generating family \mathcal{CG}_S , this translates into the following vanishing statement. Let S be a noetherian finite dimensional scheme. Let $\pi : C \rightarrow S$ be a smooth curve. Let $P = f_! \Sigma^\infty \mathbb{M} \in \mathcal{DG}_S$ (i.e., $f : V \rightarrow S$ étale, $\mathbb{M} \in \mathcal{M}_1^{\text{pure}}(V)$). Then

$$(\mathcal{V}_n(P)) \quad \forall n < 0, \mathbf{DA}(S)(M_S(C), P[n]) = 0$$

and

$$(\mathcal{V}_n^\perp(P)) \quad \forall n < 0, \mathbf{DA}(S)(M_S^\perp(C)[-1], P[n]) = 0$$

To prove this, we study the long exact sequence

$$(\mathcal{E}) \quad \dots \rightarrow \mathbf{DA}(S)(M_S(\pi_0(C/S)), P[n]) \rightarrow \mathbf{DA}(S)(M_S(C), P[n]) \rightarrow \mathbf{DA}(S)(M_S^\perp(C), P[n]) \rightarrow \dots$$

with particular attention for $n = 0$.

By localisation and Proposition 2.2, we can assume that S is reduced. By Zariski's main theorem, there exists a factorisation $f = \bar{f} \circ j$ with $\bar{f} : \bar{V} \rightarrow S$ finite and $j : V \rightarrow \bar{V}$ everywhere dense open immersion; we can assume \bar{V} is reduced as well. Combining this with the (\bar{f}^*, \bar{f}_*) adjunction, the $\text{Ex}_!^*$ isomorphism and Proposition 2.2, we get a long exact sequence

$$\begin{array}{ccc} \dots \longrightarrow \mathbf{DA}(\bar{V})(M_{\bar{V}}(\pi_0(C_{\bar{V}}/\bar{V})), j_! \Sigma^\infty \mathbb{M}[n]) & \longrightarrow & \mathbf{DA}(\bar{V})(M_{\bar{V}}(C_{\bar{V}}), j_! \Sigma^\infty \mathbb{M}[n]) \\ & & \downarrow \\ & \longleftarrow & \mathbf{DA}(\bar{V})(M_{\bar{V}}^\perp(\bar{V}), j_! \Sigma^\infty \mathbb{M}[n]) \end{array}$$

This shows we can assume $f = j$, an everywhere dense open immersion. We write $i : Z \rightarrow S$ for the complementary reduced closed immersion.

For the rest of the proof, we look separately at the three types of pure Deligne 1-motives. We want to prove (\mathcal{V}_n) and (\mathcal{V}_n^\perp) by induction on the dimension of S . In each case, to treat the case of $\dim(S) = 0$, we reduced immediately to the case of $\mathbf{Spec}(k)$ for k a field, we use the \mathcal{DG}_k family of generators instead of \mathcal{CG}_k , and we apply Proposition 4.21. We are thus left with the induction step.

Let \mathbb{M} be $[L \rightarrow 0]$ with L a lattice on V . We prove (\mathcal{V}) and (\mathcal{V}^\perp) by induction on the dimension of S .

Let $l : W \rightarrow S$ an everywhere dense open immersion with $W \subset V$ and $k : Y \rightarrow S$ the complementary reduced closed immersion. Then by localisation we have short exact sequences

$$\mathbf{DA}(S)(M_S(C), l_! l^* P[n]) \rightarrow \mathbf{DA}(S)(M_S(C), P[n]) \rightarrow \mathbf{DA}(S)(M_S(C), k_* k^* P[n])$$

and

$$\mathbf{DA}(S)(M_S^\perp(C)[-1], l_! l^* P[n]) \rightarrow \mathbf{DA}(S)(M_S^\perp(C)[-1], P[n]) \rightarrow \mathbf{DA}(S)(M_S^\perp(C)[-1], k_* k^* P[n])$$

and in both cases the right term vanishes for $n < 0$ by the (k^*, k_*) -adjunction and the induction hypothesis (since $\dim(Z) < \dim(S)$). This means we can replace P with

$$l_! l^* P \simeq l_! l^* j_! \Sigma^\infty \mathbb{M} \simeq (W \rightarrow S)_!(W \rightarrow V)^* \Sigma^\infty \mathbb{M} \simeq (W \rightarrow S)_! \Sigma^\infty \mathbb{M}_W$$

where we have used the $\text{Ex}_!^*$ isomorphism and Corollary 2.2. In other words, we can replace the dense open V by any smaller dense open W .

Using this reduction, we can assume V to be normal. This allows us by Lemma A.2 to write $\Sigma^\infty \mathbb{M}$ as a direct factor of $e_* \mathbb{Q}$ for a finite étale morphism $e : T \rightarrow V$. Applying Zariski's main theorem to the morphism $j \circ e : T \rightarrow S$ and adjunction, we reduce to the case $P = \mathbb{Q}_V$. The point of this reduction is that P then extends to a motive on S , namely \mathbb{Q}_S . By localisation, we have exact sequences

$$\mathbf{DA}(S)(M_S(C), i_* \mathbb{Q}[n-1]) \rightarrow \mathbf{DA}(S)(M_S(C), j_! \mathbb{Q}[n]) \rightarrow \mathbf{DA}(S)(M_S(C), \mathbb{Q}[n])$$

and

$$\mathbf{DA}(S)(M_S^\perp(C)[-1], i_* \mathbb{Q}[n-1]) \rightarrow \mathbf{DA}(S)(M_S^\perp(C)[-1], j_! \mathbb{Q}[n]) \rightarrow \mathbf{DA}(S)(M_S^\perp(C)[-1], \mathbb{Q}[n]),$$

and in both cases the left term vanishes for $n < 0$ by adjunction and induction. This means we can assume $V = S$.

After all these reductions, the long exact sequence (\mathcal{E}) can be written as

$$\dots \rightarrow \mathbf{DA}(S)(M_S(\pi_0(C/S)), \mathbb{Q}[n]) \rightarrow \mathbf{DA}(S)(M_S(C), \mathbb{Q}[n]) \rightarrow \mathbf{DA}(S)(M_S^\perp(C), \mathbb{Q}[n]) \rightarrow \dots$$

By adjunction and Proposition B.5 (i), we get $\mathbf{DA}(S)(M_S(C), \mathbb{Q}[n]) = 0$ for $n < 0$. By Proposition B.5 (iv) applied to $C \rightarrow \pi_0(C/S)$, which is smooth with geometrically connected fibers, we have $\mathbf{DA}(S)(M_S(\pi_0(C/S)), \mathbb{Q}) \simeq \mathbf{DA}(S)(M_S(C), \mathbb{Q})$ and $\mathbf{DA}(S)(M_S(\pi_0(C/S)), \mathbb{Q}[1]) \simeq \mathbf{DA}(S)(M_S(C), \mathbb{Q}[1])$ (to be precise, one has to apply étale descent because $\pi_0(C/S)$ is only an algebraic space). Together, this shows (\mathcal{V}_n) and (\mathcal{V}_n^\perp) for all $n < 0$.

Let now \mathbb{M} be of the form $[0 \rightarrow T]$ with T a torus on V . The proof is quite similar to the lattice case.

As in the proof for a lattice, we can replace the dense open V by any smaller dense open. Again, this lets us assume that V is normal, hence reduce to a permutation torus, then finally to $T = \mathbb{G}_m$. Then $\Sigma^\infty \mathbb{M} \simeq \mathbb{Q}_V(1)$ extends to a motive on S , namely $\mathbb{Q}_S(1)$. By localisation, we have distinguished triangles

$$\mathbf{DA}(S)(M_S(C), i_*\mathbb{Q}(1)[n-1]) \rightarrow \mathbf{DA}(S)(M_S(C), j_!\mathbb{Q}(1)[n]) \rightarrow \mathbf{DA}(S)(M_S(C), \mathbb{Q}(1)[n])$$

and

$$\mathbf{DA}(S)(M_S^\perp(C)[-1], i_*\mathbb{Q}(1)[n]) \rightarrow \mathbf{DA}(S)(M_S^\perp(C)[-1], j_!\mathbb{Q}(1)[n]) \rightarrow \mathbf{DA}(S)(M_S^\perp(C)[-1], \mathbb{Q}(1)[n]),$$

and in both cases the left term vanishes for $n < 0$ by adjunction and induction. This means we can assume $V = S$.

After these reductions, we have a long exact sequence

$$\dots \rightarrow \mathbf{DA}(S)(M_S(\pi_0(C/S)), \mathbb{Q}(1)[n]) \rightarrow \mathbf{DA}(S)(M_S(C), \mathbb{Q}(1)[n]) \rightarrow \mathbf{DA}(S)(M_S^\perp(C), \mathbb{Q}(1)[n]) \rightarrow \dots$$

By adjunction and Proposition B.6 (i), we have

$$\mathbf{DA}(S)(M_S(C), \mathbb{Q}(1)[n]) = 0 \text{ and } \mathbf{DA}(S)(M_S(\pi_0(C/S)), \mathbb{Q}(1)[n]) = 0 \text{ for all } n \leq 0.$$

The long exact sequence then implies (\mathcal{V}_n) and (\mathcal{V}_n^\perp) for all $n < 0$.

Let \mathbb{M} finally be of the form $[0 \rightarrow A]$ with A an abelian scheme on V . As in the two previous cases, we can replace the dense open V by any smaller dense open. Using [Kat99, Theorem 11] and continuity, this lets us assume that there exists a smooth projective curve $f : D \rightarrow V$ together with a section $s : V \rightarrow D$ such that the $\Sigma^\infty[0 \rightarrow A]$ is a direct factor of $\Sigma^\infty[0 \rightarrow \text{Jac}(D/V)]$. In the following, we replace A by $\text{Jac}(D/V)$.

Unlike in the two previous cases, we cannot ensure that the curve D extends to a smooth projective curve over S , so we have to work a little around this. From Corollary 4.1, we have an isomorphism $f_!\mathbb{Q}_D \simeq \mathbb{Q}_V \oplus \Sigma^\infty \text{Jac}(D/V)_\mathbb{Q} \oplus \mathbb{Q}_V(1)[2]$; hence $\Sigma^\infty \mathbb{M} \simeq \Sigma^\infty \text{Jac}(D/V)_\mathbb{Q}[-1]$ is a direct factor of $f_!\mathbb{Q}_D[-1]$. By relative purity, we have $f_!\mathbb{Q}_D[-1] \simeq f_!\mathbb{Q}_D(1)[1]$.

We apply Nagata's theorem [Nag63] [Con07] to compactify f over S : there exists an open immersion $\bar{j} : D \rightarrow \bar{D}$ and a proper morphism $\bar{f} : \bar{D} \rightarrow S$ with $j \circ f = \bar{f} \circ \bar{j}$. Write $\bar{i} : Y \rightarrow \bar{D}$ for the complementary closed immersion; note that because f was proper over V , we can choose the compactification \bar{D} so that Y lies entirely over Z . This implies that $j_!f_! \simeq \bar{j}_!\bar{f}_! \simeq \bar{f}_*\bar{j}_!$; hence $j_!f_!\mathbb{Q}_D(1)[1] \simeq \bar{f}_*\bar{j}_!\mathbb{Q}_D(1)[1]$. The motive $\bar{j}_!\mathbb{Q}_D(1)[1]$ extends to a motive on \bar{D} , namely $\mathbb{Q}_{\bar{D}}(1)[1]$. By localisation, we have a commutative diagram

$$\begin{array}{ccc}
(\mathcal{L}) & \mathbf{DA}(\bar{D})(M_{\bar{D}}(\pi_0(C_{\bar{D}}/\bar{D})), \bar{i}_*\mathbb{Q}(1)[n]) & \longrightarrow \mathbf{DA}(\bar{D})(M_{\bar{D}}(C_{\bar{D}}), \bar{i}_*\mathbb{Q}(1)[n]) \\
& \downarrow & \downarrow \\
& \mathbf{DA}(\bar{D})(M_{\bar{D}}(\pi_0(C_{\bar{D}}/\bar{D})), \bar{j}_!\mathbb{Q}(1)[n+1]) & \longrightarrow \mathbf{DA}(\bar{D})(M_{\bar{D}}(C_{\bar{D}}), \bar{j}_!\mathbb{Q}(1)[n+1]) \\
& \downarrow & \downarrow \\
& \mathbf{DA}(\bar{D})(M_{\bar{D}}(\pi_0(C_{\bar{D}}/\bar{D})), \mathbb{Q}(1)[n+1]) & \longrightarrow \mathbf{DA}(\bar{D})(M_{\bar{D}}(C_{\bar{D}}), \mathbb{Q}(1)[n+1]).
\end{array}$$

and for $n < 0$, the groups on the top and bottom line vanish by the $((-)_\# , (-)^*)$ adjunction and Proposition B.6 (i) (applying étale descent for \mathbf{DA} to get around the fact that $\pi_0(C/S)$ is only an algebraic space). Using that $j_!\Sigma^\infty \mathbb{M}$ is a direct factor of $\bar{f}_*\bar{j}_!\mathbb{Q}_D(1)[1]$, this establishes (\mathcal{V}_n) for all

$n < 0$. Plugging this back in the sequence (\mathcal{E}) , we also get (\mathcal{V}_n^\perp) for $n < -1$. However for $n = -1$, we cannot conclude directly; rather, from (\mathcal{E}) we get an exact sequence

$$0 \rightarrow \mathbf{DA}(S)(M_S^\perp(C)[-1], j_! \Sigma^\infty \mathbb{M}[-1]) \rightarrow \mathbf{DA}(S)(M_S(\pi_0(C/S)), j_! \Sigma^\infty \mathbb{M}) \rightarrow \mathbf{DA}(S)(M_S(C), j_! \Sigma^\infty \mathbb{M})$$

and we have to show that the last morphism is injective. Because $\Sigma^\infty A[-1]$ is a direct factor of $f_* \mathbb{Q}_D(1)[1]$, it suffices to show that the same morphism for $f_* \bar{j}_! \mathbb{Q}_D(1)[1]$, namely

$$\mathbf{DA}(S)(M_S(\pi_0(C/S)), \bar{f}_* \bar{j}_! \mathbb{Q}_D(1)[1]) \rightarrow \mathbf{DA}(S)(M_S(C), \bar{f}_* \bar{j}_! \mathbb{Q}_D(1)[1]),$$

is injective. By adjunction, this translates to a question in $\mathbf{DA}(\bar{D})$. Specializing diagram (\mathcal{L}) for $n = 0$, we get

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ \mathbf{DA}(\bar{D})(M_{\bar{D}}(\pi_0(C_{\bar{D}}/\bar{D})), \bar{j}_! \mathbb{Q}(1)[1]) & \rightarrow & \mathbf{DA}(\bar{D})(M_{\bar{D}}(C_{\bar{D}}), \bar{j}_! \mathbb{Q}(1)[1]) \\ \downarrow & & \downarrow \\ \mathbf{DA}(\bar{D})(M_{\bar{D}}(\pi_0(C_{\bar{D}}/\bar{D})), \mathbb{Q}(1)[1]) & \longrightarrow & \mathbf{DA}(\bar{D})(M_{\bar{D}}(C_{\bar{D}}), \mathbb{Q}(1)[1]). \end{array}$$

which shows that we only need to show the injectivity of

$$\mathbf{DA}(\bar{D})(M_{\bar{D}}(\pi_0(C_{\bar{D}}/\bar{D})), \mathbb{Q}(1)[1]) \rightarrow \mathbf{DA}(\bar{D})(M_{\bar{D}}(C_{\bar{D}}), \mathbb{Q}(1)[1]).$$

Applying cohomological h -descent for \mathbf{DA} to a proper regular hypercovering of \bar{D} (constructed using alterations after a further reduction to \bar{D} of finite type over a Dedekind ring as in the proof of Proposition B.5 (iv)), we can assume \bar{D} to be regular. Then the morphism above can be identified using Proposition B.6 (ii) with the natural morphism

$$\mathcal{O}^\times(\pi_0(C_{\bar{D}}/\bar{D})) \otimes \mathbb{Q} \rightarrow \mathcal{O}^\times(C_{\bar{D}}) \otimes \mathbb{Q}.$$

Since the morphism $C_{\bar{D}} \rightarrow \pi_0(C_{\bar{D}}/\bar{D})$ is surjective and $\pi_0(C_{\bar{D}}/\bar{D})$ is reduced, the induced morphism on global functions is injective. This completes the proof of the injectivity, hence the proof of (\mathcal{V}^\perp) in the missing case $n = -1$. This finishes the proof of the theorem. \square

Corollary 4.23. *Let S be a noetherian finite dimensional scheme. Let G be a smooth commutative group scheme with connected fibers. Then the motive $\Sigma^\infty G[-1]$ lies in $\mathbf{MM}_1(S)$.*

Proof. By noetherian induction localisation, we can assume S reduced and it is enough to show that there exists $j : U \rightarrow S$ dense open immersion such that $j_! \Sigma^\infty G_U[-1]$ is in $\mathbf{MM}_1(S)$. We can also assume S to be irreducible; let η be its generic point and $h : \eta^{\text{perf}} \rightarrow \eta$ a perfect closure. Then $G_{\bar{\eta}}$ is a smooth commutative connected algebraic group over a perfect field, hence there exists an exact sequence

$$0 \rightarrow U \rightarrow G_{\bar{\eta}} \rightarrow H \rightarrow 0$$

with U unipotent connected and H a semi-abelian variety. Since $\bar{\eta}$ is perfect, the motive $\Sigma^\infty U \otimes \mathbb{Q}$ is trivial (apply [AEWH15, Lemma 7.4.5] to a composition series), hence the morphism $h^* \Sigma^\infty G_{\bar{\eta}} \simeq \Sigma^\infty G_{\bar{\eta}, \mathbb{Q}} \rightarrow \Sigma^\infty H_{\bar{\eta}, \mathbb{Q}}$ is an isomorphism. Moreover, the motive $h_* \Sigma^\infty H_{\bar{\eta}, \mathbb{Q}}$ is isomorphic to a motive of the form $\Sigma^\infty H'_{\bar{\eta}, \mathbb{Q}}$ with H' a semi-abelian variety (this holds up to a p -isogeny, hence rationally), so we get by the separation property of $\mathbf{DA}(-)$ an isomorphism $\Sigma^\infty G_{\bar{\eta}, \mathbb{Q}} \simeq \Sigma^\infty H'_{\bar{\eta}, \mathbb{Q}}$. By spreading out arguments, we can arrange for such an isomorphism to hold over a dense open set U of S . We then have $j_! \Sigma^\infty G_{U, \mathbb{Q}}[-1] \simeq j_! \Sigma^\infty H'_{U, \mathbb{Q}}[-1]$ and this last motive is in $\mathbf{MM}_1(S)$ by Theorem 4.22. \square

In the course of the proof (in the lattice case), we also established the vanishing statements necessary to prove

Lemma 4.24. *Let S be a noetherian finite-dimensional scheme. Let $e : U \rightarrow S$ be an étale morphism. Then $e_{\sharp} \mathbb{Q} \in \mathbf{MM}_0(S)$.*

Using the same strategy as in the proof of the abelian scheme case (reduction to Jacobian, extension of the curve), one can also prove the following related vanishing result.

Proposition 4.25. *Let S be a noetherian finite-dimensional scheme. Let $e : U \rightarrow S$ be an étale morphism and A/U be an abelian scheme. Then for all $n \in \mathbb{Z}$, we have*

$$\mathbf{DA}(S)(\mathbb{Q}, e_{\sharp} \Sigma^{\infty} A(-1)[n]) = 0$$

We deduce an additional compatibility relation between the motivic t -structures on 0 and 1-motives.

Corollary 4.26. *Let S be a noetherian finite-dimensional scheme. The functor $\omega^1 : (\mathbf{DA}^1(S), t_{\mathbf{MM}}^1) \rightarrow (\mathbf{DA}^0(S), t_{\mathbf{MM}}^0)$ is t -exact.*

Proof. The functor $\omega^0 : \mathbf{DA}^1(S) \rightarrow \mathbf{DA}^0(S)$, defined as the restriction of ω^0 to $\mathbf{DA}^1(S)$, is the right adjoint to the inclusion $\mathbf{DA}_0(S) \rightarrow \mathbf{DA}^1(S)$. This inclusion is t -positive by looking at generators, which implies that its right adjoint ω^0 is t -negative.

It remains to show ω^0 is t -positive. By Lemma 3.5, ω^0 commutes with small sums. It is thus enough to show that a family of compact generators of $\mathbf{DA}^1(S)$ is sent to t -positive objects. By Proposition 4.8, $\mathbf{DA}^1(S)$ is compactly generated by $\mathcal{DG}_S(-1)$. Let $e : U \rightarrow S$ be an étale morphism and $\mathbb{M} = [L \rightarrow G] \in \mathcal{M}_1(U)$, and let $e_{\sharp}(\Sigma^{\infty} \mathbb{M})(-1) \in \mathcal{DG}_S(-1)$. We have to be careful because ω^0 and $e_{\sharp} \simeq e_!$ do not commute in general and we cannot apply directly Proposition 3.9 (iii). However, we have distinguished triangles

$$e_{\sharp} \Sigma^{\infty} T(-1) \rightarrow e_{\sharp} \Sigma^{\infty} \mathbb{M}(-1) \rightarrow e_{\sharp} \Sigma^{\infty} W_{\geq -1} \mathbb{M}(-1) \xrightarrow{+}$$

and

$$e_{\sharp} \Sigma^{\infty} A(-1) \rightarrow e_{\sharp} \Sigma^{\infty} W_{\geq -1} \mathbb{M}(-1) \rightarrow e_{\sharp} \Sigma^{\infty} L(-1) \xrightarrow{+}.$$

The motive $(e_{\sharp} \Sigma^{\infty} L)(-1)$ is in $\mathbf{DA}_0(S)(-1)$, which by Corollary 3.8 (iii) implies that its ω^0 vanishes. Let us show that we have $\omega^0(e_{\sharp} \Sigma^{\infty} A(-1)) \simeq 0$. Using the generating family, we have to show that, for all $f : W \rightarrow S$ étale and all $n \in \mathbb{Z}$, we have $\mathbf{DA}(S)(f_{\sharp} \mathbb{Q}[-n], e_{\sharp} \Sigma^{\infty} A(-1)) = 0$. By adjunction, the Ex_{\sharp}^* isomorphism and Proposition 2.2, we can assume $f = \mathrm{id}$ and apply Proposition 4.25.

All together, this means that $\omega^0(e_{\sharp} \Sigma^{\infty} \mathbb{M}(-1)) \simeq \omega^0(e_{\sharp} \Sigma^{\infty} T(-1)) \simeq e_{\sharp} \Sigma^{\infty} X_*(T)$ (Proposition 3.9 (ii)) which is t -positive for $t_{\mathbf{MM},0}(S)$. This completes the proof. \square

Notice that at this point we do not know if the motivic t -structures restricts to compact objects. A step in that direction is the following result.

Corollary 4.27. *Let S be a noetherian finite-dimensional scheme. Any compact object in either $\mathbf{DM}_0(S)$, $\mathbf{DM}^1(S)$ or $\mathbf{DM}_1(S)$ is bounded for the motivic t -structure, i.e., it has only finitely many non-zero homology objects.*

Proof. Starting from the fact that we have a family of compact generators in the heart, this follows from the structure of compact objects in a compactly generated triangulated category [Nee01, Lemma 4.4.5] by an induction argument. \square

The proof of the following result is also very similar to the proof of Theorem 4.22, hence we include it here.

Proposition 4.28. *Let S be a noetherian finite-dimensional quasi-excellent scheme. The t -structures $t_{\mathbf{MM},0}(S)$, $t_{\mathbf{MM},1}(S)$ and $t_{\mathbf{MM}}^1(S)$ are non-degenerate.*

Proof. Since $t_{\mathbf{MM}}^1 = t_{\mathbf{MM},1}(-1)$, it is enough to treat the cases of $t_{\mathbf{MM},0}$ and $t_{\mathbf{MM},1}$. These t -structures are defined as generated t -structures. By [Ayo07b, Proposition 2.1.73], to show that a t -structure of the form $t(\mathcal{G})$ for a family of compact objects \mathcal{G} is non-degenerate, it is enough to check that $\mathcal{T} = \ll \mathcal{G} \gg$ and that for $A \in \mathcal{G}$, there exists an integer $d_A \geq 0$ such that for all $B \in \mathcal{G}$, $\mathrm{Hom}(A, B[n]) = 0$ for $n \geq d_A$.

Let us check these conditions for $t_{\mathbf{MM},0}$, using the generating family $\mathcal{G}_0 = \{e_{\sharp} \mathbb{Q} | e : U \rightarrow S \text{ étale}\}$. By definition of $\mathbf{DA}_0(S)$, we have $\mathbf{DA}_0(S) = \ll \mathcal{G}_0 \gg$. Let $e : U \rightarrow S$ and $h : V \rightarrow S$ be étale morphisms. We will prove that

$$\forall n > \dim(S), \mathbf{DA}(S)(e_{\sharp} \mathbb{Q}, h_{\sharp} \mathbb{Q}[n]) = 0.$$

By the (e_{\sharp}, e^*) adjunction, we can assume $e = \mathrm{id}$. Using Zariski's main theorem, we compactify h into $h = \bar{h} \circ j$ with $j : V \rightarrow \bar{V}$ dense open immersion and $\bar{h} : \bar{V} \rightarrow S$ finite. Using the (\bar{h}^*, \bar{h}_*)

adjunction, we see that we can assume $h = j$ dense open immersion. Notice that through these reductions, the dimension of the base does not increase. Write $i : Z \rightarrow S$ for the complementary closed immersion to j . We have $\dim(Z) \leq \dim(S) - 1$. By localisation and adjunction, we have an exact sequence

$$\mathbf{DA}(Z)(\mathbb{Q}, \mathbb{Q}[n-1]) \rightarrow \mathbf{DA}(S)(\mathbb{Q}, j_! \mathbb{Q}[n]) \rightarrow \mathbf{DA}(S)(\mathbb{Q}, \mathbb{Q}[n])$$

The two outer group vanish because of Proposition B.3 (noticing that $n-1 > \dim(S)-1 \geq \dim(Z)$), and this completes the proof that $t_{\mathbf{MM},0}$ is non-degenerate.

For $t_{\mathbf{MM},1}$, we require a variant of [Ayo07b, Proposition 2.1.73], which follows from the same argument.

Lemma 4.29. *Let \mathcal{T} a compactly generated category and $\mathcal{G}, \mathcal{G}'$ two families of compact objects. We assume that $t(\mathcal{G}) = t(\mathcal{G}')$ and $\mathcal{T} = \ll \mathcal{G} \gg = \ll \mathcal{G}' \gg$. Assume that for all $A \in \mathcal{G}$ there exists an integer $d_A \geq 0$ such that for all $B \in \mathcal{G}'$, $\mathrm{Hom}(A, B[n]) = 0$ for $n \geq d_A$. \square*

We apply this lemma to $\mathcal{T} = \mathbf{DA}_1(S)$, $\mathcal{G} = \mathcal{CG}_S$, $\mathcal{G}' = \mathcal{JG}_S$. We will prove that

$$\forall M \in \mathcal{CG}_S, \forall N \in \mathcal{JG}_S, \forall n > \dim(S) + 4, \mathbf{DA}(S)(M, N[n]) = 0.$$

Let us first concentrate on M . By definition, M takes the form $M_S(C)$ or $M_S^\perp(C)[-1]$ with $\pi : C \rightarrow S$ smooth curve. In the first case, we have

$$\mathbf{DA}(S)(M_S(C), N[n]) \simeq \mathbf{DA}(C)(\mathbb{Q}_S, \pi^* N[n])$$

with $\dim(C) \leq \dim(S) + 1$ and $\pi^* N \in \mathcal{JG}_C$. In the second case, we have an exact sequence

$$\mathbf{DA}(\pi_0(C/S))(\mathbb{Q}_{\pi_0(C/S)}, \pi_0(f)^* N[n+2]) \rightarrow \mathbf{DA}(S)(M_S^\perp(C)[-1], N[n]) \rightarrow \mathbf{DA}(C)(\mathbb{Q}_C, f^* N[n+1]) \rightarrow \dots$$

with $\dim(C) \leq \dim(S) + 1$, $\dim(\pi_0(C/S)) \leq \dim(S)$ and $f^* N \in \mathcal{JG}_C$ and $\pi_0(f)^* N \in \mathcal{JG}_{\pi_0(C/S)}$. Put together, we see that it is enough to show that for all S (including étale algebraic spaces over schemes), we have

$$\forall N \in \mathcal{JG}_S, \forall n > \dim(S) + 3, \mathbf{DA}(S)(\mathbb{Q}_S, N[n]) = 0.$$

We now go into the case distinction for \mathcal{JG}_S . Let $e : U \rightarrow S$ étale morphism. The motive N is of one of the following forms: $e_{\sharp} \mathbb{Q}$, $e_{\sharp} \mathbb{Q}(1)$ or $e_{\sharp} \mathrm{Jac}(C/U)[-1]$ for $\pi : C \rightarrow U$ a smooth projective curve. By Zariski's main theorem, localisation and adjunction, we can assume $e = j$ open immersion (this does not change the dimension). In the first two cases, we apply the same argument as for $t_{\mathbf{MM},0}$: by localisation, we can assume $e = \mathrm{id}$ and then apply Proposition B.3. Let us focus on the Jacobian case. We write $\mathrm{Jac}(C/U)[-1]$ as direct factor of $M_S(C)[-1]$ by Corollary 4.1, then compactify $j \circ \pi = \bar{\pi} \circ \bar{j}$ with $\bar{j} : C \rightarrow \bar{C}$ dense open immersion and $\bar{\pi} : \bar{C} \rightarrow S$ using Zariski's main theorem. Writing $\bar{i} : Z \rightarrow \bar{C}$ for the complementary closed immersion to \bar{j} and using localisation and relative purity, we have an exact sequence

$$\mathbf{DA}(Z)(\mathbb{Q}_Z, \mathbb{Q}(1)[n]) \rightarrow \mathbf{DA}(S)(\mathbb{Q}_S, j_{\sharp} M_S(C)[-1]) \rightarrow \mathbf{DA}(\bar{C})(\mathbb{Q}, \mathbb{Q}(1)[n+1]).$$

We have $\dim(Z), \dim(\bar{C}) \leq \dim(S) + 1$, hence the two outer groups vanish for $n > \dim(S) + 3$ by Proposition B.3. This completes the proof that $t_{\mathbf{MM},1}$ is non-degenerate. \square

Finally, we compute more precisely the morphisms between Deligne 1-motives over a regular base.

Theorem 4.30. *Let S be a regular scheme, $\mathbb{M}_1, \mathbb{M}_2 \in \mathcal{M}_1(S)$, $n \in \mathbb{Z}$. Then*

$$\mathbf{DA}(S)(\Sigma^\infty \mathbb{M}_1, \Sigma^\infty \mathbb{M}_2[n]) \simeq \begin{cases} 0, & n < 0 \\ \mathcal{M}_1(S)(\mathbb{M}_1, \mathbb{M}_2), & n = 0 \\ 0, & n \geq 3. \end{cases}$$

In particular, the functor $\Sigma^\infty : \mathcal{M}_1(S) \rightarrow \mathbf{MM}_1(S)$ is fully faithful.

Proof. By considering the connected components, we reduce to the case where S is irreducible. The idea of the proof is that in the range we are considering everything happens at the generic point η . Let $j : U \rightarrow S$ closed immersion with $U \neq \emptyset$. The restriction functor $j^* : \mathcal{M}_1(S) \rightarrow \mathcal{M}_1(U)$ is fully faithful by Proposition A.11. Moreover the category $\mathcal{M}_1(\eta)$ is the 2-colimit of the $\mathcal{M}_1(U)$ for U running through all non-empty open sets of S by Proposition A.10. This implies that $\mathcal{M}_1(S)(\mathbb{M}_1, \mathbb{M}_2) \simeq \mathcal{M}_1(\eta)(\eta^* \mathbb{M}_1, \eta^* \mathbb{M}_2)$. On the $\mathbf{DA}(-)$ side, by continuity and Proposition 2.2,

we have that $\mathbf{DA}(\eta)(\eta^*\Sigma^\infty\mathbb{M}_1, \eta^*\Sigma^\infty\mathbb{M}_2[n]) \simeq \operatorname{Colim}_{U \neq \emptyset} \mathbf{DA}(U)(j^*\Sigma^\infty\mathbb{M}_1, j^*\Sigma^\infty\mathbb{M}_2[n])$. Furthermore, by Proposition 4.21, we have an isomorphism

$$\mathbf{DA}(\eta)(\Sigma^\infty\eta^*\mathbb{M}_1, \Sigma^\infty\eta^*\mathbb{M}_2[n]) \simeq \operatorname{Ext}_{\mathcal{M}_1(\eta)}^n(\mathbb{M}_1, \mathbb{M}_2) \stackrel{n \neq 0,1}{\simeq} 0$$

for $n \neq 0, 1$.

Putting everything together, we see that the statement of the proposition follows from the claim that $j^* : \mathbf{DA}(S)(\Sigma^\infty\mathbb{M}_1, \Sigma^\infty\mathbb{M}_2[n]) \rightarrow \mathbf{DA}(U)(j^*\Sigma^\infty\mathbb{M}_1, j^*\Sigma^\infty\mathbb{M}_2[n])$ is bijective for $n \neq 1, 2$. Write $i : Z \rightarrow S$ for the reduced complementary closed immersion of U in S . Consider the localisation exact sequence

$$\begin{array}{ccccc} \dots & \longrightarrow & \mathbf{DA}(Z)(i^*\Sigma^\infty\mathbb{M}_1, i^!\Sigma^\infty\mathbb{M}_2[n]) & \longrightarrow & \mathbf{DA}(S)(\Sigma^\infty\mathbb{M}_1, \Sigma^\infty\mathbb{M}_2[n]) \\ & & & & \downarrow j^* \\ \dots & \longleftarrow & \mathbf{DA}(Z)(i^*\Sigma^\infty\mathbb{M}_1, i^!\Sigma^\infty\mathbb{M}_2[n+1]) & \longleftarrow & \mathbf{DA}(U)(j^*\Sigma^\infty\mathbb{M}_1, j^*\Sigma^\infty\mathbb{M}_2[n]) \end{array}$$

We have to prove the vanishing of $\mathbf{DA}(Z)(i^*\Sigma^\infty\mathbb{M}_1, i^!\Sigma^\infty\mathbb{M}_2[n+1])$ for $n \neq 2$. By Proposition 2.2, we have $i^*\Sigma^\infty\mathbb{M}_1 \simeq \Sigma^\infty\mathbb{M}_{1,Z}$. Stratifying Z by regular constructible subschemes and applying further localisations, we can reduce to the case where Z is also regular of some codimension $1+e$ with $e \geq 0$. By absolute purity, we then have $i^!\Sigma^\infty\mathbb{M}_2[n+1] \simeq i^*\Sigma^\infty\mathbb{M}_2(-1-e)[n-1-2e] \simeq \Sigma^\infty\mathbb{M}_{2,Z}(-1-e)[n-1-2e]$. We know from Corollary 2.13 that the motive $\Sigma^\infty\mathbb{M}_{1,Z}(-1)$ lies in $\mathbf{DA}^1(S)$, hence we have an isomorphism

$$\mathbf{DA}(Z)(\Sigma^\infty\mathbb{M}_{1,Z}, \Sigma^\infty\mathbb{M}_{2,Z}(-1-e)[n-1-2e]) \simeq \mathbf{DA}(Z)(\Sigma^\infty\mathbb{M}_{1,Z}(-1), \omega^1(\Sigma^\infty\mathbb{M}_{2,Z}(-1)(-1-e)[n-1-2e]))$$

The motive $\Sigma^\infty\mathbb{M}_{2,Z}(-1)$ is cohomological, so by Corollary 3.8 the group on the right hand side vanishes unless $e = 0$. If $e = 0$, we have further $\omega^1(\Sigma^\infty\mathbb{M}_{2,Z}(-1)(-1)) \simeq \omega^0(\Sigma^\infty\mathbb{M}_{2,Z}(-1))(-1)$. This motive was computed in Proposition 3.9 (iii) and we get

$$\omega^0(\Sigma^\infty\mathbb{M}_{2,Z}(-1))(-1) \simeq \Sigma^\infty X_*(W_{-2}\mathbb{M}_{2,Z})_{\mathbb{Q}}(-1).$$

To sum up, we have reduced to show that for S regular, $\mathbb{M} \in \mathcal{M}_1(S)$ and L lattice over S , the morphism group $\mathbf{DA}(S)(\Sigma^\infty\mathbb{M}, \Sigma^\infty L_{\mathbb{Q}}[n-1])$ is 0 for $n \neq 2$. Since S is normal, the motive $\Sigma^\infty L_{\mathbb{Q}}$ is a direct factor of $e_*\mathbb{Q}$ for $e : T \rightarrow S$ finite étale (Lemma A.2). By adjunction, we are then reduced to the case $L = \mathbb{Z}$. Write $\mathbb{M} = [N \rightarrow G]$ with N a lattice and G an abelian-by-torus scheme. We have a distinguished triangle

$$\Sigma^\infty[0 \rightarrow G] \rightarrow \Sigma^\infty\mathbb{M} \rightarrow \Sigma^\infty[N \rightarrow 0] \xrightarrow{+}$$

which shows that we can treat separately the cases $\mathbb{M} = [N \rightarrow 0]$ and $\mathbb{M} = [0 \rightarrow G]$.

In the case $\mathbb{M} = [N \rightarrow 0]$, we again write N as a direct factor of a permutation lattice, which implies that $\Sigma^\infty\mathbb{M}$ is a direct factor of $e'_*\mathbb{Q}$ with $e' : T' \rightarrow S$ finite étale. By the (h_*, h^*) adjunction, we are then reduced to a computation of weight zero motivic cohomology on a regular scheme, which vanishes exactly for $n \neq 2$ by Propositions B.2 and B.5.

In the second case, we have $\Sigma^\infty\mathbb{M} = \Sigma^\infty G_{\mathbb{Q}}[-1]$, which by [AHPL14, Theorem 3.3] is a direct factor of $M_S(G)$. We are then done using the $((G \rightarrow S)_\#, (G \rightarrow S)^*)$ adjunction and Propositions B.2 and B.5. \square

Remark 4.31. Assuming S is a regular scheme, this still leaves open the determination of the morphisms groups $\mathbf{DA}(S)(\Sigma^\infty\mathbb{M}_1, \Sigma^\infty\mathbb{M}_2[1])$ and $\mathbf{DA}(S)(\Sigma^\infty\mathbb{M}_1, \Sigma^\infty\mathbb{M}_2[2])$. One can show that they do not always coincide with the Yoneda Ext-groups in the exact category $\mathcal{M}_1(S)$; for instance for $S = \mathbb{P}_{\mathbb{C}}^1$, $\mathcal{M}_1 = [\mathbb{Z} \rightarrow 0]$ and $\mathcal{M}_2 = [0 \rightarrow \mathbb{G}_m]$, the second Yoneda Ext group $\operatorname{YExt}_{\mathcal{M}_1(S)}^2([\mathbb{Z} \rightarrow 0], [0 \rightarrow \mathbb{G}_m])$ vanishes (this follows from arguments of [Org04, Proposition 3.2.4] together with the fact that any abelian scheme over $\mathbb{P}_{\mathbb{C}}^1$ is constant, i.e., comes from an abelian variety over \mathbb{C}) while

$$\mathbf{DA}(\mathbb{P}_{\mathbb{C}}^1)(\Sigma^\infty[\mathbb{Z} \rightarrow 0], \Sigma^\infty[0 \rightarrow \mathbb{G}_m][2]) \simeq \mathbf{DA}(\mathbb{P}_{\mathbb{C}}^1)(\mathbb{Q}, \Sigma^\infty\mathbb{G}_m \otimes \mathbb{Q}[1]) \simeq \mathbb{Q}.$$

APPENDIX A. DELIGNE 1-MOTIVES

We gather necessary results on Deligne 1-motives [Del74, §10] over general base schemes which we could not find in the literature. Useful references besides Deligne's original work are [Jos09], [BVK, Appendix C].

A.1. Definitions.

Definition A.1. Let S be a scheme. We say that a group scheme G/S is

- (i) *discrete* if it is étale locally constant finitely generated.
- (ii) a *lattice* if it is discrete and torsion free.
- (iii) a *abelian-by-torus scheme* if it is semi-abelian of locally constant toric rank (hence an extension of a torus by an abelian scheme by [FC90, 2.11], in a unique way).

Before we come to the definition of Deligne 1-motives, let us discuss a recurrent technical point about lattices and tori over general schemes. In general, it is not the case that a discrete group scheme is isotrivial in the étale topology. However, we have the following useful lemma.

Lemma A.2. *Let S be a locally noetherian, geometrically unibranch scheme. Let L be a lattice over S (resp. T be a torus over S).*

- (i) *L (resp. T) is isotrivial, i.e., it becomes split after passing to a finite étale cover of S .*
- (ii) *The sheaf $L \otimes \mathbb{Q} \in \mathbf{Sh}(\mathbf{Sm}/S)$ (resp. $T \otimes \mathbb{Q} \in \mathbf{Sh}(\mathbf{Sm}/S)$) is a direct factor of the sheaf $f_*\mathbb{Q}$ (resp. $f_*(\mathbb{G}_m \otimes \mathbb{Q})$) for $f : V \rightarrow S$ a finite étale cover.*

Proof. Point (i) for lattices follows from the discussion in [SGA70, Exp. X 6.2]. For tori, it is precisely [SGA70, Exp. X Théorème 5.16].

We now prove Point (ii). Let L be a lattice over S . By (i), we can find a finite étale cover $g : V \rightarrow S$ such that g^*L is split, say $g^*L \simeq \mathbb{Z}^r$. Because g is finite étale, L becomes a direct factor of g_*g^*L after inverting $\deg(f)$ by a transfer argument. We thus have $L \otimes \mathbb{Q}$ direct factor of $g_*g^*L \otimes \mathbb{Q} \simeq g_*\mathbb{Q}^r$. Write $f : V^r \rightarrow S$ for the coproduct of r copies of g . Then $g_*\mathbb{Q}^r \simeq f_*\mathbb{Q}$. This concludes the proof of (ii) for lattices. The case of tori follows by the same argument. \square

Definition A.3. Let S be a scheme. A 2-term complex of commutative S -group schemes

$$M = [L \xrightarrow{0} \overset{-1}{G}]$$

is called a *Deligne 1-motive* over S if L is a lattice and G is an abelian-by-torus scheme. A morphism of Deligne 1-motives is a morphism of complexes of group schemes, or equivalently a morphism of complex of representable sheaves on $(\mathbf{Sm}/S)_{\text{ét}}$. We denote by $\mathcal{M}_1(S, \mathbb{Z})$ the category of Deligne 1-motives. It is a pseudo-abelian additive category, with biproducts induced by fiber products of S -group schemes.

A Deligne 1-motive $M = [L \rightarrow G]$ comes with a 3-term functorial weight filtration, defined as

$$\begin{aligned} W_{-2}M &= [0 \rightarrow T] \\ W_{-1}M &= [0 \rightarrow G] \\ W_0M &= M. \end{aligned}$$

Notation A.4. Let $f : [L \rightarrow G] \rightarrow [L' \rightarrow G']$ be a morphism of Deligne 1-motives. We use the notation f_L, f_G, f_A, f_T for the induced maps $\text{Gr}_0^W f : L \rightarrow L', W_0f : G \rightarrow G', \text{Gr}_1^W f : A \rightarrow A', \text{Gr}_2^W f : T \rightarrow T'$.

We have a basic contravariant functoriality:

Definition A.5. Let $f : S' \rightarrow S$ be any morphism of schemes. Then pullback of S -group schemes along f induces an additive functor

$$f^* : \mathcal{M}_1(S, \mathbb{Z}) \rightarrow \mathcal{M}_1(S', \mathbb{Z}).$$

We are not so much interested in 1-motives per se as in the objects they define in the derived category of sheaves with rational coefficients.

Lemma A.6. *Any morphism in $\mathcal{M}_1(S, \mathbb{Z})$ which induces a quasi-isomorphism of complexes of abelian sheaves on $(\mathbf{Sm}/S)_{\text{ét}}$ is an isomorphism.*

Proof. Let $f = (f_L, f_G) : [L_1 \rightarrow G_1] \rightarrow [L_2 \rightarrow G_2]$ be a quasi-isomorphism of complexes of sheaves. By a diagram chase, this is equivalent to $\text{Ker}(f_L) \simeq \text{Ker}(f_G)$ and $\text{Coker}(f_L) \simeq \text{Coker}(f_G)$. Since $\text{Ker}(f_L)$ is locally constant finitely generated free and $\text{Ker}(f_G)$ is a group scheme whose identity component is semi-abelian and with finite π_0 , they must be both trivial. Similarly, $\text{Coker}(f_L)$ is discrete and $\text{Coker}(f_G)$ has connected fibers, so they must be both trivial. Hence f is an isomorphism. \square

We can consequently think of $\mathcal{M}_1(S, \mathbb{Z})$ as a full subcategory of $D(\mathbf{Cpl}(\mathbf{Sh}((Sm/S)_{\text{ét}}, \mathbb{Z})))$.

Definition A.7. Let S be a noetherian scheme. We write $\mathcal{M}_1(S)$ for the idempotent completion of the \mathbb{Q} -linear category $\mathcal{M}_1(S, \mathbb{Z}) \otimes \mathbb{Q}$. We say that a morphism in $\mathcal{M}_1(S)$ is integral if it comes from $\mathcal{M}_1(S, \mathbb{Z})$. For $f : S' \rightarrow S$ morphism of schemes, we still write f^* for the induced additive functor $\mathcal{M}(S) \rightarrow \mathcal{M}(S')$.

By the results above, we can and do think of $\mathcal{M}_1(S)$ as a full subcategory of $D(\mathbf{Cpl}(\mathbf{Sh}((Sm/S)_{\text{ét}}, \mathbb{Q})))$. In practice, the idempotent completion in the definition does not affect anything that we do in this paper, and we will allow ourselves statements of the form “Let $\mathbb{M} = [L \rightarrow G] \otimes \mathbb{Q}$ be an object in $\mathcal{M}_1(S)$ ” without spelling out the immediate reduction to that case.

A.2. Continuity and smoothness. We think of Deligne 1-motives as “1-motivic local systems” over the base S . The results in this section, which have classical analogues for local systems/lisse sheaves, justify in part this intuition.

We start with a lemma about discrete group schemes.

Lemma A.8. *Let S be a locally noetherian japanese scheme, η its scheme of generic points. Then the category of discrete group schemes on η is the 2-colimit of the categories of discrete group schemes on dense open subschemes of S . The same statement holds for the category of lattices.*

Proof. The statement is equivalent to the following results.

- (i) For L/η discrete group scheme, there exists $U \subset S$ dense open and L'/U discrete such that $L \simeq \eta^* L'$. Moreover, if L is a lattice, one can choose L' be a lattice as well.
- (ii) For $U \subset S$ dense open, $L, L'/U$ discrete, we have

$$\mathrm{Hom}(\eta^* L, \eta^* L') \simeq \mathrm{Colim}_{V \subset U} \mathrm{Hom}((V \rightarrow U)^* L, (V \rightarrow U)^* L').$$

We first make some reductions which apply both to (i) and (ii). By the topological invariance of the étale site, we can assume S to be reduced. Since S is locally noetherian japanese and reduced, the normal locus of S is open [Gro65, Proposition 6.13.2]. So any small enough open set U in S is normal, and in particular geometrically unibranch. By the discussion in [SGA70, Exp. X 6.2], discrete group schemes on geometrically unibranch schemes are split by finite étale covers. Moreover, for any small enough open set U the set of connected components (open by local noetherianness) of U and of η coincide. We can thus reduce to the case where η is connected (i.e., S irreducible).

We prove (i). Since η itself is normal, there is a finite étale Galois cover $\tilde{\eta}/\eta$ such that $L_{\tilde{\eta}}$ is constant. In other words, L corresponds to a representation ρ of $\mathrm{Gal}(\tilde{\eta}/\eta)$ on a finitely generated abelian group F . By [Gro66, Théorème 8.8.2, Théorème 8.10.5] and [Gro67, Théorème 17.7.8] there exists a $U \subset S$ dense open and \tilde{U}/U finite étale such that $\tilde{U} \times_U \eta \simeq \tilde{\eta}$. Up to shrinking U , one can assume it to be normal. By [Gro66, Théorème 8.8.2] applied to the finite group $\mathrm{Gal}(\tilde{\eta}/\eta)$, up to shrinking U one can assume that $\mathrm{Aut}(\tilde{U}/U) \simeq \mathrm{Gal}(\tilde{\eta}/\eta)$ (in particular \tilde{U}/U is Galois). Then the representation of $\mathrm{Gal}(\tilde{U}/U)$ on F corresponding to ρ via this isomorphism defines a discrete group scheme L'/U such that $L \simeq \eta^* L'$ as required. The addendum about lattices follows from the construction, i.e., L' is a lattice if L is.

We now prove (ii). Let $U \subset S$ dense open, $L, L'/U$ discrete group schemes. We can shrink U and assume it is normal. Let \tilde{V}/V be a finite étale Galois covering trivializing L and L' . We thus get two finitely generated abelian groups F, F' with representations ρ, ρ' of $\mathrm{Gal}(\tilde{V}/V)$. Let $\tilde{\eta} := \tilde{V} \times_V \eta$. Then $\tilde{\eta}/\eta$ is Galois with $G := \mathrm{Gal}(\tilde{V}/V) \simeq \mathrm{Gal}(\tilde{\eta}/\eta)$. Then the system in the right-hand side of (ii) is constant and both sides of (ii) are in bijection with $\mathrm{Hom}_G(\rho, \rho')$. This concludes the proof. \square

Remark A.9. It is not clear to the author how to extend this result to a more general continuity result for discrete group schemes on a projective limit of schemes with affine transition morphisms.

We deduce from this a continuity result for Deligne 1-motives.

Proposition A.10. *Let S be a locally noetherian japanese scheme, η its scheme of generic points. Then the category $\mathcal{M}_1(\eta, \mathbb{Z})$ (resp. $\mathcal{M}_1(\eta)$) is the 2-colimit of the categories $\mathcal{M}_1(U, \mathbb{Z})$ (resp. $\mathcal{M}_1(U)$) for all dense opens $U \subset S$.*

Proof. The case of $\mathcal{M}_1(-)$ follows directly from the one of $\mathcal{M}_1(-, \mathbb{Z})$. We have to show that

- (i) for all $M \in \mathcal{M}_1(\eta, \mathbb{Z})$, there exists $U \subset S$ dense open and $M' \in \mathcal{M}_1(U, \mathbb{Z})$ such that $M \simeq \eta^* M'$, and that
- (ii) for all $U \subset S$ dense open and all $M, N \in \mathcal{M}_1(U, \mathbb{Z})$:

$$\mathcal{M}_1(\eta, \mathbb{Z})(\eta^* M, \eta^* N) \simeq \operatorname{Colim}_{V \subset U} \mathcal{M}_1(V, \mathbb{Z})((V \rightarrow U)^* M, (V \rightarrow U)^* N).$$

We prove (i). Let $M = [L \rightarrow G] \in \mathcal{M}_1(\eta, \mathbb{Z})$ with the extension $0 \rightarrow T \rightarrow G \rightarrow A \rightarrow 0$.

By [Gro66, Théorème 8.8.2.(ii), Scholie 8.8.3, Théorème 8.10.5.(xii)] and [Gro67, Proposition 17.7.8], we can find an $U \subset S$ and a smooth group scheme G'/U such that $G \simeq G' \times_U \eta$. Recall that an abelian scheme is by definition a smooth proper group scheme with connected fibers, hence by [Gro66, Théorème 8.8.2.(ii), Scholie 8.8.3, Théorème 8.10.5.(xii)] and [Gro67, Proposition 17.7.8], we can shrink U and find an abelian scheme A'/U such that $A \simeq A' \times_U \eta$. By Lemma A.8 and the duality between lattices and tori, we can shrink U and assume that there exists a discrete group scheme L' and a torus T' over U such that $L \simeq L' \times_U \eta$ and $T \simeq T' \times_U \eta$.

We have spread out the pure pieces of M , it remains to glue them together. By [Gro66, Théorème 8.8.2.(i)], up to shrinking U , we have morphisms $A' \rightarrow G' \rightarrow T'$ which restrict to the extension defining G . By a standard argument based on [Gro66, Théorème 8.10.5], up to shrinking U , this is in fact an exact sequence of group schemes. Finally, we have to spread out the morphism $L \rightarrow G$. This can be done by the same Galois descent argument as in the end of the proof of Lemma A.8.

Let us now prove (ii). In $\mathcal{M}_1(-, \mathbb{Z})$, the components of a morphism are morphisms of (group) schemes. It is enough to spread them out one by one because the resulting diagram will commute by schematic density of η in S . We have treated morphisms of discrete group schemes in Lemma A.8. The case of morphisms of semi-abelian schemes (which are in particular of finite presentation) is a direct application of [Gro66, Théorème 8.8.2.(i)]. \square

When the base scheme is noetherian excellent and reduced (resp. normal), we can say more.

Proposition A.11. *Let S be a noetherian excellent scheme, $i : \eta \rightarrow S$ its scheme of generic points.*

- (i) *Suppose S reduced. Then the pullback functor $\eta^* : \mathcal{M}_1(S, \mathbb{Z}) \rightarrow \mathcal{M}_1(\eta, \mathbb{Z})$ (resp. $\eta^* : \mathcal{M}_1(S) \rightarrow \mathcal{M}_1(\eta)$) is faithful.*
- (ii) *Suppose moreover that S is normal. Then η^* is fully faithful.*

Proof. Let us prove (i). By Proposition A.10 this is equivalent to the faithfulness of the functor j^* for all $j : U \rightarrow V$ dense open subsets. It is enough to show faithfulness of j^* separately for morphisms of discrete group schemes and semi-abelian schemes, and in both cases it follows from the "reduced to separated" uniqueness criterion [Gro60, Lemme 7.2.2.1].

We now prove (ii). By Proposition A.10, it is enough to prove fullness for the functor j^* for all $j : U \rightarrow V$ dense opens. Let $M = [L \xrightarrow{u} G]$, $M' = [L' \xrightarrow{u'} G'] \in \mathcal{M}_1(V, \mathbb{Z})$ and $f_U = (f_U^L, f_U^G) : j^* M \rightarrow j^* M'$. First, using the normality of V and [SGA03, Exposé I Corollaire 10.3], the morphism f_U^L extends uniquely to a morphism $f^L : L \rightarrow L'$. Second, using the normality of V and [FC90, Proposition 2.7], the morphism f_U^G extends uniquely to a morphism $f^G : G \rightarrow G'$. The uniqueness ensures that (f^L, f^G) is a morphism $M \rightarrow M'$ which extends f_U . \square

A.3. Pushforward and Weil restriction. Let $g : S' \rightarrow S$ be a finite étale morphism. We can define a pushforward functor $g_* : \mathcal{M}_1(S') \rightarrow \mathcal{M}_1(S)$ using Weil restriction of scalars. We also include some results for finite flat morphisms. Recall the following definition.

Definition A.12. Let $g : S' \rightarrow S$ be a morphism of schemes and X/S' be a S' -scheme. The *Weil restriction* $R_g X$ is the presheaf of sets on Sch/S defined for any S -scheme U by:

$$R_g X(U) = X(U \times_S S')$$

If X/S' is a commutative group scheme (or more generally an fppf sheaf of abelian groups on Sch/S), then $R_g X$ is naturally an fppf sheaf of abelian groups on Sch/S . Moreover, the formation of R_g is functorial and compatible with base change. We summarize results on the representability of $R_g X$ from the litterature.

Proposition A.13. *Let $g : S' \rightarrow S$ be a morphism of schemes and X/S' be a S' -scheme.*

- (i) [Ols06, Theorem 1.5] Assume that g is proper flat of finite presentation. Then $R_g X$ is representable by an algebraic space (We note that we will only need the case g finite flat, which is presumably easier, but I could not find a reference).
- (ii) [BLR90, 7.6/5] Assume that g is finite flat. Then if X is smooth (resp. of finite presentation) then $R_g X$ (which exists at least as an algebraic space by (i)) is smooth (resp. of finite presentation).
- (iii) [BLR90, 7.6/5] Assume that g is finite étale. Then if X is proper then $R_g X$ (which exists at least as an algebraic space by (i)) is proper.
- (iv) [BLR90, 7.6/2] Let $h : X \rightarrow Y$ be a closed immersion of S' -schemes. Then $R_g h : R_g X \rightarrow R_g Y$ is a closed immersion of presheaves. As a corollary, if X/S is affine, then $R_g X$ is representable by an affine scheme.

We now use the results above to analyse Weil restriction of pure 1-motives. We are spared from having to consider algebraic spaces by the following result.

Proposition A.14. *Let $g : S' \rightarrow S$ be finite flat.*

- (1) *Let T/S' be a torus (resp. L/S' be a lattice). Then $R_g T$ is a torus (resp. $R_g L$ is a lattice).*
- (2) *Let A/S' be an abelian scheme. Assume that g is étale. Then $R_g A$ is an abelian scheme.*

Proof. By Proposition A.13 (iv), we know that $R_g T$ and $R_g L$ are representable by affine S' -group schemes. Moreover, because of the compatibility with base change and étale descent, it is enough to consider the case of a split torus and a constant lattice over S' , in which case the Weil restrictions are directly seen to be a split torus or a constant lattice over S .

By Proposition A.13 (i)-(iii), we know that $R_g A$ is representable by a proper smooth algebraic group space over S . By [FC90, Theorem 1.9], this implies that $R_g A$ is an abelian scheme. \square

Now we tackle the case of semi-abelian schemes.

Lemma A.15. *Let $g : S' \rightarrow S$ be a morphism of schemes.*

- (i) *When restricted to fppf sheaves of abelian groups, the functor R_g is left exact.*
- (ii) *Assume that g is finite flat. Let $f : G \rightarrow H$ be a smooth and surjective morphism between commutative group schemes of finite presentation. Then the morphism of algebraic group spaces $R_g f : R_g G \rightarrow R_g H$ is smooth and surjective.*
- (iii) *Assume g is finite flat. Let $0 \rightarrow G' \xrightarrow{i} G \xrightarrow{p} G'' \rightarrow 0$ be an exact sequence of smooth commutative S -group schemes with $G \rightarrow G''$ flat (and hence smooth). The sequence $0 \rightarrow R_g G' \rightarrow R_g G \rightarrow R_g G'' \rightarrow 0$ is exact.*

Proof. Point (i) is clear from the definition. We turn to point (ii). The fact that $R_g f$ is smooth follows from the infinitesimal criterion of smoothness (and does not require that we are working with group schemes). The surjectivity can be tested point-wise on S , so that by compability of R_g with base change we can assume that S is the spectrum of a field k . Surjectivity is a geometric property, so that we can assume k to be algebraically closed as well. We then have to check the surjectivity of the induced map $R_g G(k) = G(S') \rightarrow R_g H(k) = H(S')$ on k -points. Since S'/k is finite flat, it is a product of finite local algebras. Surjectivity then follows from the surjectivity of f , the fact that k is algebraically closed, and the formal smoothness of f . Note that if g is finite étale, we do not need f smooth.

For (iii), it is enough to check that $R_g G'$ is the scheme-theoretic kernel of $R_g p$ and that $R_g p$ is an fppf morphism. The first assertion follows from (i), and the second from (ii). \square

Proposition A.16. *Let $g : S' \rightarrow S$ be finite étale and G/S be an abelian-by-torus scheme. Then $R_g G$ is an abelian-by-torus scheme.*

Proof. The result follows directly from Proposition A.14 and Lemma A.15 (iii). \square

Definition A.17. Let $g : S' \rightarrow S$ be a finite étale morphism. We define the Weil restriction of a Deligne 1-motive $M = [L \xrightarrow{u} G] \in \mathcal{M}_1^{\mathbb{Z}}(S')$ as $R_g M = [R_g L \xrightarrow{R_g u} R_g G]$ which is in $\mathcal{M}_1(S)$ by Propositions A.14 and A.16. This induces a functor

$$g_* : \mathcal{M}_1^{\mathbb{Z}}(S') \rightarrow \mathcal{M}_1^{\mathbb{Z}}(S)$$

which is by construction a right adjoint to g^* .

APPENDIX B. MOTIVIC COHOMOLOGY IN DEGREES $(*, \leq 1)$

We gather here some computations of rational motivic cohomology groups which are used at various places in this paper. Most of the following is present, explicitly or implicitly, in [Ayo14a, §11] and in the K-theoretic interpretation of rational motivic cohomology provided by the comparison with Beilinson motives [CD, §14].

Notation B.1. Let S be a noetherian finite dimensional scheme. For $p, q \in \mathbb{Z}$, we write $H_{\mathcal{M}}^{p,q}(S) := \mathbf{DA}(S)(\mathbb{Q}_S, \mathbb{Q}_S(q)[p])$.

Proposition B.2. [Ayo14a, Proposition 11.1 (b)] *Let S be a noetherian finite-dimensional scheme. For all $w < 0$ and $n \in \mathbb{Z}$, we have $H_{\mathcal{M}}^{n,w}(S) \simeq 0$.*

Proposition B.3. *Let S be a noetherian finite dimensional quasi-excellent scheme. For all $i \in \mathbb{N}$ and $n > \dim(S) + 2i$, we have $H_{\mathcal{M}}^{n,i}(S) \simeq 0$.*

Proof. The group $H_{\mathcal{M}}^{n+2i,i}(S) \simeq \mathbf{DA}(S)(\mathbb{Q}[n], \mathbb{Q}(i)[2i])$ is a direct factor of $\mathbf{DA}(S)(\mathbb{Q}[n], \sum_{i \in \mathbb{Z}} \mathbb{Q}(i)[2i])$. By Theorem [CD, 16.2.18], this group is isomorphic to $\mathbf{DM}_{\mathbb{B}}(S)(\mathbb{Q}[n], \sum_{i \in \mathbb{Z}} \mathbb{Q}(i)[2i])$. By Corollary [CD, 14.2.17], we have $\sum_{i \in \mathbb{Z}} \mathbb{Q}(i)[2i] \simeq \mathrm{KGL}_{\mathbb{Q},S}$ where the last object is the \mathbb{Q} -localisation of the motivic spectrum KGL_S . This implies that

$$\mathbf{DM}_{\mathbb{B}}(S)(\mathbb{Q}[n], \mathrm{KGL}_{\mathbb{Q},S}) \simeq \mathbf{SH}(S)(\Sigma^n \Sigma_T^\infty(S_+), \mathrm{KGL}) \otimes \mathbb{Q}.$$

By [Cis13, Théorème 2.20], this last group is isomorphic to $\mathrm{KH}_n(S) \otimes \mathbb{Q}$, where KH is homotopy-invariant K -theory. Finally, by the main step in the proof of [Kel14, Theorem 3.5], under our hypotheses on S (including quasi-excellent), the group $\mathrm{KH}_n(S) \otimes \mathbb{Q}$ vanishes for $n < -\dim(S)$. This completes the proof. \square

Remark B.4. We only need this result for $i = 0, 1$ and it is likely that there is a non- K -theoretic proof in those cases, combining results below on $H_{\mathcal{M}}^{n,0}, H_{\mathcal{M}}^{n,1}$ of regular schemes with an ingenious use of resolution of singularities by alterations as in the proof of [Kel14, Theorem 3.5].

Let S be a scheme. Then we have $D(\mathrm{Sm}/S)(\mathbb{Q}_S, \mathbb{Q}_S) \simeq \mathbb{Q}^{\pi_0(S)}$ (with $\pi_0(S)$ the set of connected components of S). This provides a morphism

$$\nu^{0,0} : \mathbb{Q}^{\pi_0(S)} \simeq D(\mathrm{Sm}/S)(\mathbb{Q}_S, \mathbb{Q}_S) \longrightarrow \mathbf{DA}(S)(\mathbb{Q}_S, \mathbb{Q}_S) = H_{\mathcal{M}}^{0,0}(S)$$

More generally, we have for all $n \in \mathbb{Z}$ a morphism

$$\nu^{n,0} : D(\mathrm{Sm}/S)(\mathbb{Q}_S, \mathbb{Q}_S[n]) \longrightarrow H_{\mathcal{M}}^{n,0}(S)$$

Proposition B.5.

- (i) *For all $n < 0$, we have $H_{\mathcal{M}}^{n,0}(S) \simeq 0$.*
- (ii) *The morphism $\nu^{0,0}$ induces an isomorphism $H_{\mathcal{M}}^{0,0}(S) \simeq \mathbb{Q}^{\pi_0(S)}$.*
- (iii) *Assume S regular. For all $n > 0$, we have $H_{\mathcal{M}}^{n,0}(S) \simeq 0$.*
- (iv) *Let $f : T \rightarrow S$ be a smooth surjective morphism with geometrically connected generic fibers. Then for all $n \in \mathbb{Z}$, we have $f^* : H_{\mathcal{M}}^{n,0}(S) \xrightarrow{\sim} H_{\mathcal{M}}^{n,0}(T)$.*

Proof. Statements (i) and (ii) are proved in [Ayo14a, Proposition 11.1 (a)]. More precisely, in [Ayo14a, Proposition 11.1 (a)], an unspecified isomorphism $H_{\mathcal{M}}^{0,0}(S) \simeq \mathbb{Q}^{\pi_0(S)}$ is constructed. Let us sketch why it corresponds to $\nu^{0,0}$ by going through the proof in loc. cit.

Step A is a reduction to rational coefficients and is irrelevant to us.

Step B consists of several reductions. First we have a reduction to the affine case via a Mayer-Vietoris sequence, and reduction to S of finite type over a Dedekind scheme D by a filtered projective limit argument. The Mayer-Vietoris sequence for the Zariski topology is also available for $D(\mathrm{Sm}/S)$ and is compatible with the one for $H_{\mathcal{M}}^{n,0}$ via the $\nu^{n,0}$. From [Gro66, Proposition 8.4.2], we see that $\pi_0(S)$ is compatible with filtered projective limits. Then there is an induction on the dimension. The case $\dim(S) = 0$ uses the comparison with $\mathbf{DM}^{\mathrm{ét}}(k)$, the cancellation theorem to reduce to a computation in the derived category of sheaves with transfers on SmCor/k . To see that this computation is compatible with the map $\nu^{0,0}$, one just needs to introduce the similar map on sheaves with transfers and look at the equivalence $\mathbf{DA}(k) \simeq \mathbf{DM}(k)$. The last part of Step B is a reduction to the normalisation. It relies on localisation and base change for finite morphisms, both of which are available for $D(\mathrm{Sm}/S)$, and compatible with the $\nu^{n,0}$ maps.

Step C settles the case of a normal positive characteristic scheme. The argument reduces to the case of a smooth \mathbb{F}_p scheme via de Jong's alterations. The main problem for the alteration argument is that the proper base change theorem does not hold in general for $D(\mathbf{Sm}/S)$, and so we do not have a priori the analogue of the long exact sequence (111) of loc. cit. However, one can first prove that $\nu^{n,0}$ is an isomorphism in the regular case, and then deduce that in the case of interest, the proper base change map is an isomorphism. We leave the details to the reader. For the last part of step C, namely the case of a smooth \mathbb{F}_p -variety X , we have $H_{\text{ét}}^0(X, \mathbb{Q}) = \mathbb{Q}_0^\pi(X)$ and $\nu^{n,0}$ is clearly an isomorphism in this case.

Step D settles the case where the fiber in characteristic 0 is non-empty. The resolution and alteration arguments have to be adapted as in step C. We are then reduced to the case S smooth over D . We have $H_{\mathcal{M}}^{n,0}(S, \mathbb{Q}) \simeq H_{\mathcal{M}}^{n,0}(S \times \mathbb{Q}, \mathbb{Q})$ by absolute purity, which is not available in $D(\mathbf{Sm}/S)$. To remedy this, we have to show that $H_{\text{ét}}^0(S, \mathbb{Q}) \simeq H_{\text{ét}}^0(S \times \mathbb{Q}, \mathbb{Q})$. This follows from the fact that since S/D is smooth, we have $\pi_0(S) \simeq \pi_0(S \times \mathbb{Q})$.

This completes the sketch of the proof that $\nu^{0,0}$ is an isomorphism.

Let us prove Statement (iii). Fix $n > 0$. We can assume that S is connected with generic point η . By the argument at the beginning of the proof of [Ayo14a, Corollaire 11.4], combining absolute purity and localisation with the vanishing of negative motivic cohomology B.2, one can deduce that for any dense open set U in S , the restriction map $H_{\mathcal{M}}^{n,0}(S) \rightarrow H_{\mathcal{M}}^{n,0}(U)$ is injective. By the continuity property of [Ayo14a, Proposition 3.20], we deduce that the restriction map $H_{\mathcal{M}}^{n,0}(S) \rightarrow H_{\mathcal{M}}^{n,0}(\eta)$ is injective. So we are reduced to the case where S is the spectrum of a field k .

By separation, we can assume that k is perfect. By [CD, Corollary 16.2.22], we reduce to compute $\mathbf{DM}(k, \mathbb{Q})(\mathbb{Q}_k, \mathbb{Q}_k[n])$. By the cancellation theorem [Voe10], we reduce to compute $\mathbf{DM}^{\text{eff}}(k, \mathbb{Q})(\mathbb{Q}_k, \mathbb{Q}_k[n])$. Since the sheaf \mathbb{Q}_k is both cofibrant and \mathbb{A}^1 -local, this coincides with the same Hom group computed in the derived category of étale sheaves over \mathbf{Sm}/S , which vanishes. This concludes the proof of (iii).

Let us prove Statement (iv). By Mayer-Vietoris, we can assume S to be affine. By a limit argument using the continuity property of \mathbf{DA} , we can then assume that S is of finite type over a Dedekind ring. Using [dJ97, Corollary 5.15] applied to the irreducible components of the normalisation of S and then iterating, we build a proper hypercovering $\pi_\bullet : \tilde{S}_\bullet \rightarrow S$ with all \tilde{S}_n regular. We pullback π_\bullet to obtain a proper hypercovering $\pi'_\bullet : \tilde{T}_\bullet \rightarrow T$. Since f is smooth, all \tilde{T}_n are regular as well. By cohomological descent for the h-topology [CD, Theorem 14.3.4], we have $\mathbb{Q}_S \simeq \pi_{\bullet*} \mathbb{Q}_{\tilde{S}_\bullet}$ and $\mathbb{Q}_T \simeq \pi'_{\bullet*} \mathbb{Q}_{\tilde{T}_\bullet}$. We deduce that $H_{\mathcal{M}}^{n,0}(S) \simeq \mathbf{DA}(\tilde{S}_\bullet)(\mathbb{Q}_{\tilde{S}_\bullet}, \mathbb{Q}_{\tilde{S}_\bullet}[n])$ and $H_{\mathcal{M}}^{n,0}(T) \simeq \mathbf{DA}(\tilde{T}_\bullet)(\mathbb{Q}_{\tilde{T}_\bullet}, \mathbb{Q}_{\tilde{T}_\bullet}[n])$. By (i), (ii) and (iii), we have for every $k, m \in \mathbb{Z}$ that $\mathbf{DA}(\tilde{S}_k)(\mathbb{Q}_{\tilde{S}_k}, \mathbb{Q}_{\tilde{S}_k}[m])$ is isomorphic to $BQ^{\pi_0(\tilde{S}_k)}$ if $m = 0$ and 0 otherwise; a similar formula holds for \tilde{T} . Now the map f and its pullbacks induce isomorphisms $\pi_0(S_k) \simeq \pi_0(T_k)$ on sets of connected components because f has geometrically connected generic fibers (a property which is itself stable by pullback). This implies the result. \square

Let S be a scheme. We have $D(\mathbf{Sm}/S)(\mathbb{Q}_S, \mathbb{G}_m \otimes \mathbb{Q}) \simeq H^0(S_{\text{ét}}, \mathbb{G}_m \otimes \mathbb{Q}) \simeq \mathcal{O}^\times(S) \otimes \mathbb{Q}$ and $D(\mathbf{Sm}/S)(\mathbb{Q}_S, \mathbb{G}_m \otimes \mathbb{Q}[1]) \simeq H^1(S_{\text{ét}}, \mathbb{G}_m \otimes \mathbb{Q}) \simeq \text{Pic}(S) \otimes \mathbb{Q}$. Combining these isomorphisms with Proposition 2.6, this induces morphisms

$$\nu^{1,1} : \mathcal{O}^\times(S) \longrightarrow H_{\mathcal{M}}^{1,1}(S)$$

and

$$\nu^{2,1} : \text{Pic}(S)_{\mathbb{Q}} \longrightarrow H_{\mathcal{M}}^{2,1}(S).$$

More generally, for any $n \in \mathbb{Z}$, we have an induced morphism

$$\nu^{n,1} : D(\mathbf{Sm}/S)(\mathbb{Q}_S, \mathbb{G}_m[n-1]) \rightarrow H_{\mathcal{M}}^{n,1}(S).$$

Proposition B.6.

- (i) For all $n \leq 0$, we have $H_{\mathcal{M}}^{n,1}(S) \simeq 0$.
- (ii) Assume S regular. The morphism $\nu^{1,1}$ induces an isomorphism $H_{\mathcal{M}}^{1,1}(S) \simeq \mathcal{O}^\times(S)_{\mathbb{Q}}$.
- (iii) Assume S regular. The morphism $\nu^{2,1}$ induces an isomorphism $H_{\mathcal{M}}^{2,1}(S) \simeq \text{Pic}(S)_{\mathbb{Q}}$.
- (iv) Assume S regular. For all $n \neq 1, 2$, we have $H_{\mathcal{M}}^{n,1}(S) \simeq 0$. We have also $D(\mathbf{Sm}/S)(\mathbb{Q}_S, \mathbb{G}_m \otimes \mathbb{Q}[n-1]) \simeq 0$, so that the morphism $\nu^{n,1}$ is an isomorphism.

Proof. Statement (i) for S regular and a weaker version of (ii) (without specifying the isomorphism) are proved in [Ayo14a, Corollaire 11.4].

To pass from (i) for S regular to a general S , we apply resolution of singularities by alterations and cohomological h -descent for a proper regular hypercovering (which induces a descent spectral sequence for $H^{n,1}(-)$). To be more precise, one has to reduce to a situation where one can apply De Jong's theorem, e.g. S of finite type over a Dedekind ring: for this, one uses Mayer-Vietoris to first reduce to S affine, and then continuity.

We revisit and precise the argument in [Ayo14a, Corollaire 11.4] to establish (ii), (iii) and (iv).

Let us first treat the case where S is the spectrum of a field. In that case, for $\neq 1$, both the source and target of $\nu^{n,1}$ are 0, so the only interesting case is $n = 1$. We have to show that the map

$$\nu_k^{1,1} : k^\times \otimes \mathbb{Q} \rightarrow H_{\mathcal{M}}^{1,1}(k)$$

is an isomorphism. By the definition of $\nu^{1,1}$, we have to show that the map

$$k^\times \otimes \mathbb{Q} \simeq \mathbf{DA}^{\text{eff}}(k)(\mathbb{Q}, \mathbb{G}_m \otimes \mathbb{Q}) \rightarrow \mathbf{DA}(k)(\mathbb{Q}, \Sigma^\infty(\mathbb{G}_m \otimes \mathbb{Q}))$$

induced by Σ^∞ is an isomorphism.

Let k^{perf} be a perfect closure of k and $h : \mathbf{Spec}(k^{\text{perf}}) \rightarrow \mathbf{Spec}(k)$ be the canonical morphism. In the diagram

$$\begin{array}{ccc} \mathbf{DA}^{\text{eff}}(k)(\mathbb{Q}, \mathbb{G}_m \otimes \mathbb{Q}) & \longrightarrow & \mathbf{DA}(k)(\mathbb{Q}, \Sigma^\infty(\mathbb{G}_m \otimes \mathbb{Q})) \\ h^* \downarrow & & \downarrow h^* \\ \mathbf{DA}^{\text{eff}}(k^{\text{perf}})(\mathbb{Q}, \mathbb{G}_m \otimes \mathbb{Q}) & \longrightarrow & \mathbf{DA}(k^{\text{perf}})(\mathbb{Q}, h^* \Sigma^\infty(\mathbb{G}_m \otimes \mathbb{Q})) \xrightarrow[\sim]{(R_h)_*} \mathbf{DA}(k)(\mathbb{Q}, \Sigma^\infty(\mathbb{G}_m \otimes \mathbb{Q})) \end{array}$$

the left square commutes because of the natural isomorphism $h^* \Sigma^\infty \simeq \Sigma^\infty h^*$. The left vertical arrow is an isomorphism because $k^\times \otimes \mathbb{Q} \simeq (k^{\text{perf}})^\times \otimes \mathbb{Q}$ (any element of k^{perf} has a power in k), and the right vertical arrow is an isomorphism by separation for \mathbf{DA} .

We are now reduced to the case k perfect. Then we can follow a familiar pattern: comparison with $\mathbf{DM}(k)$ using [AHPL14, Theorem 2.8, Proposition 2.10], then with $\mathbf{DM}^{\text{eff}}(k)$ using Voevodsky's cancellation theorem (this is where we need k perfect), and finally the classical computation of weight one effective motivic cohomology [MVW06b, Lecture 4].

We now do the general case. We can assume S connected, hence integral. Let $j : U \rightarrow S$ be a non-empty open set, Z its closed complement. We stratify $Z = Z_0 \subset Z_1 \subset \dots \subset Z_k = \emptyset$ in such a way that for all i , the scheme $(Z_i \setminus Z_{i+1})_{\text{red}}$ is regular and in such a way that $(Z \setminus Z_1)$ contains all points of codimension 1 of Z in S . Then by applying inductively localisation, absolute purity (for the regular pair $(S, (Z_i \setminus Z_{i+1})_{\text{red}})$) and the vanishing result Proposition B.5 (i) and (ii) we see that

- the map $u^{0,0} : \mathbb{Q}^{\pi_0(Z \setminus Z_1)} \rightarrow H_{\mathcal{M}}^{0,0}(Z \setminus Z_1)$ is an isomorphism,
- the pullback map $H_{\mathcal{M}}^{n,1}(S) \rightarrow H_{\mathcal{M}}^{n,1}(U)$ is an isomorphism for $\neq 1, 2$, and
- there is a short exact sequence

$$0 \rightarrow H_{\mathcal{M}}^{1,1}(S) \rightarrow H_{\mathcal{M}}^{1,1}(U) \rightarrow H_{\mathcal{M}}^{0,0}(Z \setminus Z_1) \rightarrow H_{\mathcal{M}}^{2,1}(S) \rightarrow H_{\mathcal{M}}^{2,1}(U) \rightarrow 0$$

Putting this together with the localisation sequence for \mathcal{O}^\times and Pic , we get a diagram

$$\begin{array}{ccccccc} 0 \rightarrow \mathcal{O}_S^\times \otimes \mathbb{Q} \rightarrow \mathcal{O}_U^\times \otimes \mathbb{Q} \xrightarrow{\text{val}} \mathbb{Q}^{\pi_0(Z \setminus Z_1)} \simeq \bigoplus_{z \in Z(1)} \mathbb{Q}[z] \rightarrow \text{Pic}(S) \otimes \mathbb{Q} \rightarrow \text{Pic}(U) \otimes \mathbb{Q} \rightarrow 0 \\ \nu_S^{1,1} \downarrow \quad \nu_U^{1,1} \downarrow \quad \quad \quad \text{(A)} \quad \nu^{0,0} \downarrow \sim \quad \quad \quad \text{(B)} \quad \nu_S^{2,1} \downarrow \quad \nu_U^{2,1} \downarrow \\ 0 \rightarrow H_{\mathcal{M}}^{1,1}(S) \rightarrow H_{\mathcal{M}}^{1,1}(U) \longrightarrow H_{\mathcal{M}}^{0,0}(Z \setminus Z_1) \longrightarrow H_{\mathcal{M}}^{2,1}(S) \longrightarrow H_{\mathcal{M}}^{2,1}(U) \longrightarrow 0. \end{array}$$

We claim that the diagram above is commutative. For the two outer diagrams, this follows from the commutation of u_S with pullbacks in Proposition 2.6.

For the commutation of diagrams (A) and (B), we have to do more work, since one arrow is defined explicitly using valuations and line bundle attached to a divisor while the other is defined via the absolute purity isomorphism. Instead of giving a long explicit computation, we prefer to see it as a special case of Déglise's machinery of "residual Riemann-Roch formulas" in [DĀI, 4.2.1, 5.5.1]; namely, take the diagram (4.2.1 b) in loc. cit. with \mathbb{E} being algebraic K -theory tensor \mathbb{Q} ,

\mathbb{F} being motivic cohomology with rational coefficients, the morphism ϕ being the Chern character, and then use that $\mathcal{O}^\times(S)_\mathbb{Q} \oplus \text{Pic}(S)_\mathbb{Q} \subset K_1(S) \otimes \mathbb{Q}$ for S regular, and that the Chern character maps coincide with the maps $\nu^{n,1}$ modulo this identification.

Passing to the limit in the previous commutative diagram over all non-empty open sets, using continuity both for motivic cohomology and for the étale cohomology of \mathbb{G}_m , we get a commutative diagram

$$\begin{array}{ccccccccc} 0 \rightarrow \mathcal{O}_S^\times \otimes \mathbb{Q} \rightarrow \kappa(S)^\times \otimes \mathbb{Q} & \xrightarrow{\text{val}} & \bigoplus_{z \in S^{(1)}} \mathbb{Q}[z] & \rightarrow & \text{Pic}(S) \otimes \mathbb{Q} & \rightarrow & \text{Pic}(\kappa(S)) & \rightarrow & 0 \\ \nu_S^{1,1} \downarrow & & \nu_U^{1,1} \downarrow & & \parallel & & \nu_S^{2,1} \downarrow & & \nu_U^{2,1} \downarrow \\ 0 \rightarrow H_{\mathcal{M}}^{1,1}(S) \rightarrow H_{\mathcal{M}}^{1,1}(\kappa(S)) & \rightarrow & \bigoplus_{z \in S^{(1)}} \mathbb{Q}[z] & \rightarrow & H_{\mathcal{M}}^{2,1}(S) & \rightarrow & H_{\mathcal{M}}^{2,1}(\kappa(S)) & \rightarrow & 0. \end{array}$$

Using the case of a base field treated above, we see that

- the group $H_{\mathcal{M}}^{n,1}(S)$ vanishes for $n \neq 1, 2$, and
- there is a short exact sequence

$$0 \rightarrow H_{\mathcal{M}}^{1,1}(S) \rightarrow \kappa(S)^\times \otimes \mathbb{Q} \xrightarrow{\text{val}} \bigoplus_{z \in S^{(1)}} \mathbb{Q}[z] \rightarrow H_{\mathcal{M}}^{2,1}(S) \rightarrow 0.$$

Using the normality (resp. regularity) of S , this implies $H_{\mathcal{M}}^{1,1}(S) \simeq \mathcal{O}(S)_\mathbb{Q}^\times$ and $H_{\mathcal{M}}^{2,1}(S) \simeq \text{Pic}(S)_\mathbb{Q}$ and finishes the proof. \square

We finish by giving an example which shows that even for weight zero motivic cohomology on normal (but not regular) schemes, the result can differ from étale cohomology.

Proposition B.7. *Let S be a normal excellent surface. Let $\pi : \tilde{S} \rightarrow S$ be a resolution of singularities of S , with $D = \pi^{-1}(p)$ simple normal crossing divisor in \tilde{S} . Let $\Gamma = (V, E)$ be the resolution graph of D . Then*

$$H_{\mathcal{M}}^{n,0}(S) \simeq \begin{cases} \mathbb{Q}, & n = 0 \\ H^1(\Gamma, \mathbb{Q}), & n = 2 \\ 0, & n \neq 0, 2 \end{cases}$$

while on the other hand

$$D(\text{Sm}/S)(\mathbb{Q}_S, \mathbb{Q}_S[n]) \simeq \begin{cases} \mathbb{Q}, & n = 0 \\ 0, & n \neq 0 \end{cases}.$$

Proof. The last statement comes from the fact that the étale cohomology of a normal scheme with \mathbb{Q} -coefficients is trivial. So we concentrate on the first. For $n \leq 0$, the result follows from B.5, so we assume $n > 0$.

We have the cartesian diagram of schemes:

$$\begin{array}{ccccc} U & \xrightarrow{\tilde{j}} & \tilde{S} & \xleftarrow{\tilde{i}} & D \\ \parallel & & \downarrow \pi & & \downarrow \pi_p \\ U & \xrightarrow{j} & S & \xleftarrow{i} & p \end{array}$$

Localization yields the long exact sequence:

$$\begin{array}{ccccccc} \mathbf{DA}(S)(\mathbb{Q}_S, \mathbb{Q}_S[n-1]) & \rightarrow & \mathbf{DA}(U)(\mathbb{Q}_U, \mathbb{Q}_U[n-1]) & \rightarrow & \mathbf{DA}(p)(\mathbb{Q}_p, i^! \mathbb{Q}_S[n]) & \rightarrow & \mathbf{DA}(S)(\mathbb{Q}_S, \mathbb{Q}_S[n]) \\ & & & & & & \downarrow \\ & & & & & & \mathbf{DA}(U)(\mathbb{Q}_U, \mathbb{Q}_U[n]) \end{array}$$

By Proposition B.5, this yields an isomorphism $\mathbf{DA}(p)(\mathbb{Q}_p, i^! \mathbb{Q}_S[n]) \simeq \mathbf{DA}(S)(\mathbb{Q}_S, \mathbb{Q}_S[n])$.

Write $\{D_v\}_{v \in V}$ for the set of irreducible components of D and p_e for the intersection points $D_v \cap D_{v'}$ for $vv' \in E$. We put $Z = \bigcup_{e \in E} \{p_e\}$ and $\dot{D} = D \setminus Z$. Write $k : \dot{D} \rightarrow E$, $l : Z \rightarrow D$. Localization gives a distinguished triangle

$$l_*(\tilde{i} \circ l)^! \mathbb{Q}_{\tilde{S}} \rightarrow \tilde{i}^! \mathbb{Q}_{\tilde{S}} \rightarrow k_*(\tilde{i} \circ k)^! \mathbb{Q}_{\tilde{S}} \xrightarrow{+}.$$

By the relative purity theorem for \mathbf{DA} (see [Ayo07a, 1.6.1] and [Ayo14a, Corollaire 3.10]) applied to the regular immersions $\tilde{i} \circ l$ and $\tilde{i} \circ k$, this triangle takes the form:

$$l_*\mathbb{Q}_Z(-2)[-4] \rightarrow \tilde{i}^!\mathbb{Q}_{\tilde{S}} \rightarrow k_*\mathbb{Q}_{\tilde{D}}(-1)[-2] \xrightarrow{+}$$

So we get the exact sequence:

$$\mathbf{DA}(\tilde{D})(\mathbb{Q}_{\tilde{D}}, \mathbb{Q}_{\tilde{D}}(-2)[n-4]) \rightarrow \mathbf{DA}(D)(\mathbb{Q}_D, \tilde{i}^!\mathbb{Q}_{\tilde{S}}[n]) \rightarrow \mathbf{DA}(Z)(\mathbb{Q}_Z, \mathbb{Q}_Z(-1)[n-2])$$

By Proposition B.2, the groups on the left and on the right are zero for all $n \in \mathbb{Z}$, so we conclude that $\mathbf{DA}(D)(\mathbb{Q}_D, \tilde{i}^!\mathbb{Q}_{\tilde{S}}[n]) = 0$ for all $n \in \mathbb{Z}$.

Now, the fact that π_U is an isomorphism, colocalization and base change for immersions (see [Ayo07a, 1.4.6]) implies that $\text{Cone}(\tilde{i}^!\mathbb{Q}_S \rightarrow \pi_{p,*}\tilde{i}^!\mathbb{Q}_{\tilde{S}}) \simeq \text{Cone}(\mathbb{Q}_p \rightarrow \pi_{p,*}\mathbb{Q}_D)$. Combining with the previous result, we get that for all $n \in \mathbb{Z}$:

$$\mathbf{DA}(S)(\mathbb{Q}_S, \mathbb{Q}_S[n]) \simeq \mathbf{DA}(p)(\mathbb{Q}_p, \text{Cone}(\mathbb{Q}_p \rightarrow \pi_{p,*}\mathbb{Q}_D)[n-1]) \simeq \mathbf{DA}(p)(\mathbb{Q}_p, \pi_{p,*}\mathbb{Q}_D[n-1])$$

(The last isomorphism comes because $n > 1$).

Using Čech descent for closed covers and Proposition B.2, it is then easy to see that this last group is isomorphic to \mathbb{Q} if $n = 0$ (note that Γ is connected by normality of S), isomorphic to $H^1(\Gamma, \mathbb{Q})$ if $n = 1$, and 0 otherwise. \square

REFERENCES

- [ABV09] Joseph Ayoub and Luca Barbieri-Viale. 1-motivic sheaves and the Albanese functor. *J. Pure Appl. Algebra*, 213(5):809–839, 2009.
- [AEWH15] Giuseppe Ancona, Stephen Enright-Ward, and Annette Huber. On the motive of a commutative algebraic group. *Documenta mathematica*, 20:807–858, 2015.
- [AHPL14] Giuseppe Ancona, Annette Huber, and Simon Pepin Lehalleur. On the relative motive of a commutative group scheme. *Algebraic Geometry, to appear. Preprint: arXiv:1409.3401*, 2014.
- [Ayo07a] Joseph Ayoub. Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique. I. *Astérisque*, (314):x+466 pp. (2008), 2007.
- [Ayo07b] Joseph Ayoub. Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique. II. *Astérisque*, (315):vi+364 pp. (2008), 2007.
- [Ayo10] Joseph Ayoub. Note sur les opérations de Grothendieck et la réalisation de Betti. *J. Inst. Math. Jussieu*, 9(2):225–263, 2010.
- [Ayo11] Joseph Ayoub. The n -motivic t -structures for $n = 0, 1$ and 2. *Adv. Math.*, 226(1):111–138, 2011.
- [Ayo14a] Joseph Ayoub. La réalisation étale et les opérations de Grothendieck. *Ann. Sci. Éc. Norm. Supér. (4)*, 47(1):1–145, 2014.
- [Ayo14b] Joseph Ayoub. L’algèbre de Hopf et le groupe de Galois motiviques d’un corps de caractéristique nulle. I. *J. Reine Angew. Math.*, 693:1–149, 2014.
- [Ayo15] Joseph Ayoub. Motifs des variétés analytiques rigides. *Mém. Soc. Math. Fr. (N.S.)*, (140-141):vi+386, 2015.
- [AZ12] Joseph Ayoub and Steven Zucker. Relative Artin motives and the reductive Borel-Serre compactification of a locally symmetric variety. *Invent. Math.*, 188(2):277–427, 2012.
- [Bei02] Alexander Beilinson. Remarks on n -motives and correspondences at the generic point. In *Motives, polylogarithms and Hodge theory, Part I (Irvine, CA, 1998)*, volume 3 of *Int. Press Lect. Ser.*, pages 35–46. Int. Press, Somerville, MA, 2002.
- [Bei12] A. Beilinson. Remarks on Grothendieck’s standard conjectures. In *Regulators*, volume 571 of *Contemp. Math.*, pages 25–32. Amer. Math. Soc., Providence, RI, 2012.
- [Bha12] Bhargav Bhatt. Annihilating the cohomology of group schemes. *Algebra Number Theory*, 6(7):1561–1577, 2012.
- [BLR90] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud. *Néron models*, volume 21 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1990.
- [Bon15] Mikhail V. Bondarko. Mixed motivic sheaves (and weights for them) exist if ‘ordinary’ mixed motives do. *Compos. Math.*, 151(5):917–956, 2015.
- [Bro09] Sylvain Brochard. Foncteur de Picard d’un champ algébrique. *Math. Ann.*, 343(3):541–602, 2009.
- [Bro14] S. Brochard. Duality for commutative group stacks. *Arxiv e-prints*, April 2014.
- [BS13] B. Bhatt and P. Scholze. The pro-étale topology for schemes. *Arxiv e-prints*, September 2013.
- [BVK] Luca Barbieri Viale and Bruno Kahn. On the derived category of 1-motives. *Astérisque (to appear)*.
- [CD] D.-C. Cisinski and F. D’Àgliste. Triangulated categories of mixed motives. *Arxiv e-prints*.
- [CD15] Denis-Charles Cisinski and Frédéric Déglise. Étale motives. *Compositio Mathematica*, pages 1–111, 10 2015.
- [Cis13] Denis-Charles Cisinski. Descente par éclatements en K -théorie invariante par homotopie. *Ann. of Math. (2)*, 177(2):425–448, 2013.

- [Con07] Brian Conrad. Deligne’s notes on Nagata compactifications. *J. Ramanujan Math. Soc.*, 22(3):205–257, 2007.
- [Dég11] F. Déglise. Modules homotopiques. *Doc. Math.*, 16:411–455, 2011.
- [Del74] Pierre Deligne. Théorie de Hodge. III. *Inst. Hautes Études Sci. Publ. Math.*, (44):5–77, 1974.
- [dJ96] A. J. de Jong. Smoothness, semi-stability and alterations. *Inst. Hautes Études Sci. Publ. Math.*, (83):51–93, 1996.
- [dJ97] A. Johan de Jong. Families of curves and alterations. *Ann. Inst. Fourier (Grenoble)*, 47(2):599–621, 1997.
- [DĀĪ] F. DĀĪglise. Orientation theory in algebraic geometry.
- [Eke90] Torsten Ekedahl. On the adic formalism. In *The Grothendieck Festschrift, Vol. II*, volume 87 of *Progr. Math.*, pages 197–218. Birkhäuser Boston, Boston, MA, 1990.
- [FC90] Gerd Faltings and Ching-Li Chai. *Degeneration of abelian varieties*, volume 22 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1990. With an appendix by David Mumford.
- [Gro60] A. Grothendieck. Éléments de géométrie algébrique. I. Le langage des schémas. *Inst. Hautes Études Sci. Publ. Math.*, (4):228, 1960.
- [Gro63] A. Grothendieck. Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. II. *Inst. Hautes Études Sci. Publ. Math.*, (17):91, 1963.
- [Gro65] A. Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II. *Inst. Hautes Études Sci. Publ. Math.*, (24):231, 1965.
- [Gro66] A. Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III. *Inst. Hautes Études Sci. Publ. Math.*, (28):255, 1966.
- [Gro67] A. Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV. *Inst. Hautes Études Sci. Publ. Math.*, (32):361, 1967.
- [Gro68] Alexander Grothendieck. Le groupe de Brauer. II. Théorie cohomologique. In *Dix Exposés sur la Cohomologie des Schémas*, pages 67–87. North-Holland, Amsterdam, 1968.
- [Gro95a] Alexander Grothendieck. Technique de descente et théorèmes d’existence en géométrie algébrique. V. Les schémas de Picard: théorèmes d’existence. In *Séminaire Bourbaki, Vol. 7*, pages Exp. No. 232, 143–161. Soc. Math. France, Paris, 1995.
- [Gro95b] Alexander Grothendieck. Technique de descente et théorèmes d’existence en géométrie algébrique. VI. Les schémas de Picard: propriétés générales. In *Séminaire Bourbaki, Vol. 7*, pages Exp. No. 236, 221–243. Soc. Math. France, Paris, 1995.
- [HK06] Annette Huber and Bruno Kahn. The slice filtration and mixed Tate motives. *Compos. Math.*, 142(4):907–936, 2006.
- [Jan94] Uwe Jannsen. Motivic sheaves and filtrations on Chow groups. In *Motives (Seattle, WA, 1991)*, volume 55 of *Proc. Sympos. Pure Math.*, pages 245–302. Amer. Math. Soc., Providence, RI, 1994.
- [Jos09] Peter Jossen. *On the arithmetic of 1-motives*. PhD thesis, Central European University Budapest, 2009.
- [Kat99] Nicholas M. Katz. Space filling curves over finite fields. *Math. Res. Lett.*, 6(5-6):613–624, 1999.
- [Kel14] Shane Kelly. Vanishing of negative K -theory in positive characteristic. *Compos. Math.*, 150(8):1425–1434, 2014.
- [Kle05] Steven L. Kleiman. The Picard scheme. *Fundamental algebraic geometry Math. Surveys Monogr.*, 123:235–321, 2005.
- [MVW06a] C. Mazza, V. Voevodsky, and C. Weibel. *Lecture Notes on Motivic Cohomology*. Clay Mathematics Monographs. American Mathematical Society, 2006.
- [MVW06b] Carlo Mazza, Vladimir Voevodsky, and Charles Weibel. *Lecture notes on motivic cohomology*, volume 2 of *Clay Mathematics Monographs*. American Mathematical Society, Providence, RI, 2006.
- [Nag63] Masayoshi Nagata. A generalization of the imbedding problem of an abstract variety in a complete variety. *J. Math. Kyoto Univ.*, 3:89–102, 1963.
- [Nee01] Amnon Neeman. *Triangulated categories*, volume 148 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2001.
- [Ols06] Martin C. Olsson. $\underline{\mathrm{Hom}}$ -stacks and restriction of scalars. *Duke Math. J.*, 134(1):139–164, 2006.
- [Org04] Fabrice Orgogozo. Isomotifs de dimension inférieure ou égale à un. *Manuscripta Math.*, 115(3):339–360, 2004.
- [Ray70] Michel Raynaud. *Faisceaux amples sur les schémas en groupes et les espaces homogènes*. Lecture Notes in Mathematics, Vol. 119. Springer-Verlag, Berlin, 1970.
- [Rom11] Matthieu Romagny. Composantes connexes et irréductibles en familles. *Manuscripta Math.*, 136(1-2):1–32, 2011.
- [SGA70] *Schémas en groupes. II: Groupes de type multiplicatif, et structure des schémas en groupes généraux*. Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3). Dirigé par M. Demazure et A. Grothendieck. Lecture Notes in Mathematics, Vol. 152. Springer-Verlag, Berlin, 1970.
- [SGA71] *Théorie des intersections et théorème de Riemann-Roch*. Lecture Notes in Mathematics, Vol. 225. Springer-Verlag, Berlin, 1971. Séminaire de Géométrie Algébrique du Bois-Marie 1966–1967 (SGA 6), Dirigé par P. Berthelot, A. Grothendieck et L. Illusie. Avec la collaboration de D. Ferrand, J. P. Jouanolou, O. Jussila, S. Kleiman, M. Raynaud et J. P. Serre.
- [SGA73] *Théorie des topos et cohomologie étale des schémas. Tome 3*. Lecture Notes in Mathematics, Vol. 305. Springer-Verlag, Berlin, 1973. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4),

- Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de P. Deligne et B. Saint-Donat.
- [SGA03] *Revêtements étales et groupe fondamental (SGA 1)*. Documents Mathématiques (Paris) [Mathematical Documents (Paris)], 3. Société Mathématique de France, Paris, 2003.
- [Sta] The Stacks Project Authors. Stacks project. <http://stacks.math.columbia.edu>.
- [Voe10] Vladimir Voevodsky. Cancellation theorem. *Doc. Math.*, (Extra volume: Andrei A. Suslin sixtieth birthday):671–685, 2010.
- [VSF00] Vladimir Voevodsky, Andrei Suslin, and Eric M. Friedlander. *Cycles, transfers, and motivic homology theories*, volume 143 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2000.

E-mail address: `simon.pepin@math.uzh.ch`